On the Theory of Matrices.

By Henry Taber.

I.—Elements of the Theory.

Introductory.

§1. Cayley, in his *Memoir on the Theory of Matrices* (Phil. Trans., 1858), defined a matrix as "a set of quantities arranged in the form of a square,"* this notion arising "from an abbreviated notation for a set of linear equations." Accordingly, Cayley laid down the laws of combination of matrices upon the basis of the combined effect of the matrices as operators of linear transformation upon a set of scalar variables or carriers. The development of the theory, as contained in Cayley's memoir, was the development of the consequences of these primary laws of combination. Before Cayley's memoir appeared, Hamilton had investigated the theory of such a symbol of operation as would convert three vectors into three linear functions of those vectors, which he called a linear vector operator. Such an operator is essentially identical with a matrix as defined by Cayley; and some of the chief points in the theory of matrices were made out by Hamilton and published in his *Lectures on Quaternions* (1852). They were, however, made out as theorems concerning linear vector operators, and developed by quaternion methods, through the effect of these operators upon vectors, and not upon the basis of the primary laws of combination above referred to. Nevertheless, Hamilton must be regarded as the originator of the theory of matrices, as he was the first to show that the symbol of a linear transformation might be made the subject-matter of a calculus. Cayley makes no reference

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* Cayley speaks also of *rectangular* matrices, and to some extent develops their theory; he even alludes to the possibility of attaching a more general meaning to the word. His memoir deals, however, almost exclusively with square matrices; and as the present paper relates exclusively to such, we shall make no further reference to other than square matrices.

to Hamilton, and was of course unaware that results essentially identical with some of his had been obtained by Hamilton; and, on the other hand, Hamilton's results related only to matrices of the third and fourth order, while Cayley's method was absolutely general. The identity of the two theories was explicitly mentioned by Clifford in a passage of his *Dynamic*, and was virtually recognized elsewhere by himself and by Tait. Sylvester carried the investigation much farther, developing the subject on the same basis as that which Cayley had adopted. Subsequent to Cayley, but previous to Sylvester, the Peirces, especially Charles Peirce, were led to the consideration of matrices from a different point of view; namely, from the investigation of linear associative algebras involving any number of linearly independent units. In this aspect, the quantities arranged in a square are looked upon as scalar coefficients of the several units or "vids" of an algebra belonging to a certain class. Finally, it must be mentioned that Hamilton and his successors made use of an interpretation of a linear vector operator which consists in regarding the operator or matrix as representing a homogeneous strain; and in this light the axes of the strain play an important part in the theory. These axes may, however, be utilized without any reference to this interpretation, the analytical definition of the axes being obvious; and I have made much use of them in the following investigation.

This paper originated in an investigation upon the development of Clifford's geometrical algebras; the consideration of the linear vector functions of these algebras led me to think of investigating the theory of matrices viewed as linear vector operators. For matrices in general these algebras furnish an instrument analogous to that which quaternions affords for the investigation of matrices of the third order. I shall, however, reserve to a subsequent paper the consideration of this particular system of algebras (of which I have obtained a tolerably complete development) and its utilization for the theory of matrices. In the present paper I regard a matrix as a linear unit operator, operating upon the linearly independent units of an algebra, without reference to any meaning of such units, or to any properties which these units may have in combination with each other; and I have in this way endeavored to develop the theory of matrices. From this point of view I am able to give a very simple proof of Cayley's "identical equation," and also of Sylvester's most important results, the law of latency, the law of nullity, and Sylvester's formula for any function of a single matrix. I have also completed the investigation of the nullity of the

factors of the identical equation (Sylvester's "corollary of the law of nullity") which Sylvester had treated only in the special case when all the latent roots of the matrix were distinct; and I have shown that in addition to nonions there are an infinity of algebras exactly analogous to quaternions. This analogy I have also extended, and it appears that every matrix is resolvable, precisely as a quaternion is, into the product of a tensor and a versor; the latter gives rise in a matrix of any order to functions analogous to the sine and cosine to which these functions reduce when $\omega = 2$; and thus I find that there exists an infinity of formulae analogous to De Moivre's. Finally, I have shown that the whole theory of matrices may be regarded as contained in the theory of Clifford's geometric algebras, i. e. in the theory of sets of quaternion algebras which are such that the units of one set are commutative with those of any other. Other results I have obtained not immediately connected with the pure theory of matrices, but having reference to the matrix viewed as a linear unit operator.

These results are contained in the second part of this paper; the first part contains only the elementary notions and theorems developed from the point of view of the matrix as an operator, and is not necessary to the understanding of the second part, at all events by one acquainted with Cayley's memoir. To this must be excepted §9, which contains an account of Charles Peirce's system of quadrate algebras and their connection with the theory of matrices.

I have in I §10 given a slight sketch of the history of the theory.

Definition of a Linear Unit Operator.

§2. The extension of any selection of ω of the units of any algebra will be termed a *ground* of order ω. A *linear unit operator* ϕ of a ground of order ω is an operator which converts each of ω linearly independent quantities in the ground into a quantity in the same ground, and which is such that

$$\phi(m\sigma + n\pi) = m(\phi\sigma) + n(\phi\pi),$$

where σ and π are any two quantities in the ground, and m and n are scalars.* Thus, if $\alpha_1, \alpha_2, \ldots . \alpha_\omega$ are ω linearly independent quantities of a ground of order ω, and if

$$\rho = x_1\alpha_1 + x_2\alpha_2 + \ldots . + x_\omega\alpha_\omega,$$

* Of course this requirement is equivalent to the requirement that ϕ shall be distributive over any sum of quantities in the ground; the equation would then follow at once for m and n positive integers, and thence very simply for m and n any scalars.

then $$\phi\rho = x_1 . \phi\alpha_1 + x_2 . \phi\alpha_2 + \dots + x_\omega . \phi\alpha_\omega ;$$

where, since the $\phi\alpha$'s are in the same ground as the α's, and these are ω in number and linearly independent, we may put

$$\begin{aligned}
\phi\alpha_1 &= a_{11}\alpha_1 + a_{21}\alpha_2 + \dots + a_{\omega 1}\alpha_\omega ,\\
\phi\alpha_2 &= a_{12}\alpha_1 + a_{22}\alpha_2 + \dots + a_{\omega 2}\alpha_\omega ,\\
&\dots\dots\dots\dots\dots\dots\\
\phi\alpha_\omega &= a_{1\omega}\alpha_1 + a_{2\omega}\alpha_2 + \dots + a_{\omega\omega}\alpha_\omega :
\end{aligned}$$

whence the *linear unit function* $\phi\rho$ becomes equal to

$$(a_{11}x_1 + a_{12}x_2 + \dots + a_{1\omega}x_\omega)\,\alpha_1 + (a_{21}x_1 + a_{22}x_2 + \dots + a_{2\omega}x_\omega)\,\alpha_2 + \dots \\ + (a_{\omega 1}x_1 + a_{\omega 2}x_2 + \dots + a_{\omega\omega}x_\omega)\,\alpha_\omega .$$

By choosing the α's as the units of the set $(x_1, x_2, \dots x_\omega)$, it may be regarded as identical with ρ, when it is evident that ϕ is the matrix

$$\left(\begin{array}{llll} a_{11} & a_{12} & \dots & a_{1\omega} \\ a_{21} & a_{22} & \dots & a_{2\omega} \\ \dots & \dots & \dots & \dots \\ a_{\omega 1} & a_{\omega 2} & \dots & a_{\omega\omega} \end{array}\right)$$

Thus, either the successive columns of the array constituting the matrix may be regarded as operating upon the successive quantities $(\alpha_1, \alpha_2, \dots \alpha_\omega)$ representing the ground, or the successive rows may be regarded as operating upon the coefficients $(x_1, x_2, \dots x_\omega)$ of these quantities.

Regarding ϕ as operating upon the set $\rho = (x_1, x_2, \dots x_\omega)$, i. e. upon the coefficients of the α's in the expression for ρ, the resulting set consisting of the coefficients in the expression for the linear unit function $\phi\rho$ may be denoted by $((\phi|x_1), (\phi|x_2), \dots (\phi|x_\omega))$; and so

$$\phi\rho = (\phi|x_1)\,\alpha_1 + (\phi|x_2)\,\alpha_2 + \dots + (\phi|x_\omega)\,\alpha_\omega .$$

In virtue of the identification of a matrix and a linear unit operator, these terms will hereafter be used interchangeably.

The coefficients of the α's in the expression for the $\phi\alpha$'s which form the columns of the matrix, will be termed its constituents.

The *converse* or *transverse* of ϕ, denoted by $\breve{\phi}$, will be defined as the matrix formed from ϕ by interchanging the rows and columns of its constituents.* Whence

* The term *transverse* was used by Cayley in his memoir to denote this function of a matrix, and has subsequently been used by Sylvester in the same sense; the term *converse* was that employed by Charles Peirce for this purpose, and the symbol of conversion or transversion employed here is also his. Hamilton used the term *conjugate* of ϕ to denote the converse of ϕ, and denoted it by ϕ'.

$$\breve{\phi}\rho = x_1 . \breve{\phi}\alpha_1 + x_2 . \breve{\phi}\alpha_2 + \ldots . + x_\omega . \breve{\phi}\alpha_\omega ,$$

where

$$\begin{aligned} \breve{\phi}\alpha_1 &= a_{11}\alpha_1 + a_{12}\alpha_2 + \ldots . + a_{1\omega}\alpha_\omega , \\ \breve{\phi}\alpha_2 &= a_{21}\alpha_1 + a_{22}\alpha_2 + \ldots . + a_{2\omega}\alpha_\omega , \\ &\text{etc.} \end{aligned}$$

The relation "converse of" is evidently reciprocal, thus $\breve{\breve{\phi}} = \phi$.

Two matrices of the same ground are equal if they have invariably the same effect upon the same quantity. Thus if ψ is the matrix

$$\left(\begin{array}{cccc} b_{11} & b_{12} & \ldots . & b_{1\omega} \\ b_{21} & b_{22} & \ldots . & b_{2\omega} \\ \ldots & \ldots & \ldots & \ldots \\ b_{\omega 1} & b_{\omega 2} & \ldots . & b_{\omega\omega} \end{array}\right)$$

and the α's still represent the ground, so that

$$\begin{aligned} \psi\alpha_1 &= b_{11}\alpha_1 + b_{21}\alpha_2 + \ldots . + b_{\omega 1}\alpha_\omega , \\ \psi\alpha_2 &= b_{12}\alpha_1 + b_{22}\alpha_2 + \ldots . + b_{\omega 2}\alpha_\omega , \\ &\text{etc.,} \end{aligned}$$

and if $\phi\rho = \psi\rho$, for any quantity in the ground, then $a_{11} = b_{11}$, $a_{12} = b_{12}$, etc. Consequently, if two matrices are equal, every constituent of the one is equal to the corresponding constituent of the other; and so an equation between matrices gives rise to ω^2 scalar equations.

In defining the linear unit operator ϕ by its effect upon ω linearly independent quantities in a certain ground of order ω, the law of multiplication of the units of the ground is not in any way involved. Whence it follows that there is only an apparent loss of generality in taking the units of the ground from any particular algebra. If the elementary units of Clifford's ω-way algebra* are chosen, any quantity in the ground is a vector in ω-dimensional space; ϕ may then be termed a linear vector operator, and has a definite geometrical signification, namely, it represents a homogeneous strain. For if ρ and σ are vectors to two parallel straight lines, then we may put $\rho = \alpha_1 + x\beta$, $\sigma = \alpha_2 + y\beta$, and by definition $\phi\rho = \phi\alpha_1 + x\phi\beta$, $\phi\sigma = \phi\alpha_2 + y\phi\beta$, so the displacement of points in space which ϕ effects is such that parallel lines remain parallel; whence, extensions of any order which are parallel remain parallel after the application of the

* Clifford's ω-way algebra is an algebra linear in the product of even order of ω "elementary units," $i_1 i_2 \ldots . i_\omega$, such that $i_r i_s = -i_s i_r$, and $i_r^2 = -1$. See this Journal, Vol. I.

strain ϕ. Without, however, determining the nature of the units of the ground, i. e. the properties its units may have in combination, two extensions belonging to the ground of order m and n respectively, where $m < n$, may be termed parallel if every quantity in the one may be represented, for some set of values of the x's, by

$$\alpha_1 + x_1\beta_1 + x_2\beta_2 + \ldots + x_m\beta_m,$$

and every quantity of the other, for some set of values of the y's, by

$$\alpha_2 + y_1\beta_1 + y_2\beta_2 + \ldots + y_m\beta_m + \ldots + y_n\beta_n,$$

the α's and β's being quantities of the ground. A homogeneous strain may then be defined as a displacement in a certain extension (namely, that of the ground) which leaves unaltered the parallelism of any two extensions of the ground. Obviously ϕ effects such a displacement of quantities.

A consequence of some interest follows from the identification of the theory of linear vector operators with the theory of matrices: as quaternions is identical with the theory of dual matrices, and thus with the theory of homogeneous strains in a plane, to every proposition concerning space of three dimensions (or of four dimensions) which can be proved by quaternions, corresponds a proposition concerning the kinematics of a plane, such that when either is proved so also is the other.

Hereafter it should be understood, that when quantities are spoken of as the operands of a matrix ϕ, they are to be regarded as in the ground pertaining to ϕ even when this is not explicitly stated.

Addition of Matrices, Multiplication by a Scalar, and Form of a Scalar.

§3. Employing the notation of the last section for ϕ and ψ, let χ denote the matrix

$$\begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} \ldots\ldots a_{1\omega} + b_{1\omega} \\ a_{21} + b_{21} & a_{22} + b_{22} \ldots\ldots a_{2\omega} + b_{2\omega} \\ \text{etc., etc.} & \end{pmatrix}$$

It is obvious from the definition of a linear unit operator that for any quantity ρ,

$$\chi\rho = \phi\rho + \psi\rho.$$

Whence this sum may be denoted by $(\phi + \psi)\rho$, giving the equation in matrices, $\phi + \psi = \chi$, or

$$\left(\begin{array}{lll} a_{11} & a_{12} \ldots\ldots & a_{1\omega} \\ a_{21} & a_{22} \ldots\ldots & a_{2\omega} \\ \text{etc.} & & \end{array}\right) + \left(\begin{array}{lll} b_{11} & b_{12} \ldots\ldots & b_{1\omega} \\ b_{21} & b_{22} \ldots\ldots & b_{2\omega} \\ \text{etc.} & & \end{array}\right) = \left(\begin{array}{lll} a_{11}+b_{11} & a_{12}+b_{12} \ldots\ldots & a_{1\omega}+b_{1\omega} \\ a_{21}+b_{21} & a_{22}+b_{22} \ldots\ldots & a_{2\omega}+b_{2\omega} \\ \text{etc.} & & \end{array}\right)$$

As a consequence of the nature of scalar addition, it follows that the addition of matrices is commutative and associative, and such that its inverse, subtraction, is determinative, or may be regarded as the direct operation performed with an inverse quantity, the negative. The negative of ϕ, which may be denoted by $(-\phi)$, is of course that matrix whose constituents are the negative of the corresponding constituents of ϕ; otherwise $\phi\rho + (-\phi)\rho$ would not be zero for all values of ρ.

Evidently the converse of the sum of two matrices is the sum of their converses. Thus:

$$\overset{\smile}{(\phi+\psi)} = \breve{\phi} + \breve{\psi}.$$

§4. From the last section it follows that if m is any positive integer,

$$(m)\left(\begin{array}{lll} a_{11} & a_{12} \ldots\ldots & a_{1\omega} \\ a_{21} & a_{22} \ldots\ldots & a_{2\omega} \\ \text{etc.} & & \end{array}\right) = \left(\begin{array}{lll} ma_{11} & ma_{12} \ldots\ldots & ma_{1\omega} \\ ma_{21} & ma_{22} \ldots\ldots & ma_{2\omega} \\ \text{etc.} & & \end{array}\right)$$

Thence it may be shown very simply that the equation holds for any scalar m. And hence, obviously, for any scalar m, $(m\phi)\rho = \phi(m\rho) = m(\phi\rho)$. If, then, n is any other scalar, for any quantity ρ in the ground we have $(m.n\phi)\rho = m(n\phi.\rho) = m(n.\phi\rho) = mn.\phi\rho = (mn.\phi)\rho$; also $(m+n)\phi.\rho = (m+n).\phi\rho = m.\phi\rho + n.\phi\rho = m\phi.\rho + n\phi.\rho = (m\phi + n\phi)\rho$; consequently, $m.n\phi = mn.\phi$ and $(m+n)\phi = m\phi + n\phi$, and therefore scalars as multipliers of matrices have all the properties they possess in combination with monomial or scalar quantities. Hence in future the combination $m.\phi\rho = m\phi.\rho$ may be denoted by $m\phi\rho$, etc.

A scalar is itself obviously a linear unit operator or matrix; but the only matrix which merely multiplies a set or quantity by a scalar m is

$$\left(\begin{array}{llll} m & 0 & 0 & \text{etc.} \\ 0 & m & 0 & \text{etc.} \\ 0 & 0 & m & \text{etc.} \\ \text{etc.} & & & \end{array}\right)$$

where all the constituents are zero except those in the principal diagonal, and

they are all equal to m. If $m=1$, this array gives the matrix form of unity; if $m=0$, it gives the matrix form of zero. It will presently be shown that the product of two matrices is also a matrix, and that the product is distributive over the operand. In conjunction with the matrix form of a scalar, this makes significant the compound ϕm, m being a scalar. For since for any quantity ρ $(\phi m)\rho = \phi(m\rho) = m(\phi\rho)$; hence $\phi m = m\phi$.

Multiplication of Matrices and its Inverse.

§5. The combination $\phi\psi\rho$ must be defined as the result of operating by the linear unit operator ϕ upon the linear vector function or quantity $\phi\rho$, i. e. $\phi\psi.\rho = \phi(\psi\rho)$. With the same notation as before for ϕ and ψ, let

$$\chi = \begin{pmatrix} \Sigma_r a_{1r} b_{r1} & \Sigma_r a_{1r} b_{r2} & \dots & \Sigma_r a_{1r} b_{r\omega} \\ \Sigma_r a_{2r} b_{r1} & \Sigma_r a_{2r} b_{r2} & \dots & \Sigma_r a_{2r} b_{r\omega} \\ \dots & \dots & \dots & \dots \\ \Sigma_r a_{\omega r} b_{r1} & \Sigma_r a_{\omega r} b_{r2} & \dots & \Sigma_r a_{\omega r} b_{r\omega} \end{pmatrix}$$

Then for any quantity ρ,

$$\begin{aligned} \phi\psi\rho &= \phi\,[\Sigma_s b_{1s} x_s . \alpha_1 + \Sigma_s b_{2s} x_s . \alpha_2 + \dots + \Sigma_s b_{\omega s} x_s . \alpha_\omega] \\ &= (a_{11} . \Sigma_s b_{1s} x_s + a_{12} . \Sigma_s b_{2s} x_s + \dots + a_{1\omega} . \Sigma_s b_{\omega s} x_s)\,\alpha_1 \\ &\quad + (a_{21} . \Sigma_s b_{1s} x_s + a_{22} . \Sigma_s b_{2s} x_s + \dots + a_{2\omega} . \Sigma_s b_{\omega s} x_s)\,\alpha_2 + \text{etc.} \\ &= \Sigma_r \Sigma_s a_{1r} b_{rs} x_s . \alpha_1 + \Sigma_r \Sigma_s a_{2r} b_{rs} x_s . \alpha_2 + \dots + \Sigma_r \Sigma_s a_{\omega r} b_{rs} x_s . \alpha_\omega \\ &= (\Sigma_r a_{1r} b_{r1} . x_1 + \Sigma_r a_{1r} b_{r2} . x_2 + \dots + \Sigma_r a_{1r} b_{r\omega} . x_\omega)\,\alpha_1 \\ &\quad + (\Sigma_r a_{2r} b_{r1} . x_1 + \Sigma_r a_{2r} b_{r2} . x_2 + \dots + \Sigma_r a_{2r} b_{r\omega} . x_\omega)\,\alpha_2 + \text{etc.} \\ &= \chi\rho. \end{aligned}$$

Whence we may put $\phi\psi = \chi$, or

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1\omega} \\ a_{21} & a_{22} & \dots & a_{2\omega} \\ \dots & \dots & \dots & \dots \\ a_{\omega 1} & a_{\omega 2} & \dots & a_{\omega\omega} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1\omega} \\ b_{21} & b_{22} & \dots & b_{2\omega} \\ \dots & \dots & \dots & \dots \\ b_{\omega 1} & b_{\omega 2} & \dots & b_{\omega\omega} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} b_{11} + a_{12} b_{21} + \dots + a_{1\omega} b_{\omega 1}, & a_{11} b_{12} + a_{12} b_{22} + \dots + a_{1\omega} b_{\omega 2}, & \text{etc.} \\ a_{21} b_{11} + a_{22} b_{21} + \dots + a_{2\omega} b_{\omega 1}, & a_{21} b_{12} + a_{22} b_{22} + \dots + a_{2\omega} b_{\omega 2}, & \text{etc.} \\ \dots & \dots & \dots \\ a_{\omega 1} b_{11} + a_{\omega 2} b_{21} + \dots + a_{\omega\omega} b_{\omega 1}, & a_{\omega 1} b_{12} + a_{\omega 2} b_{22} + \dots + a_{\omega\omega} b_{\omega\omega}, & \text{etc.} \end{pmatrix}$$

i. e. the product of the two matrices ϕ and ψ in the order named is formed as

follows: The constituent of the r^{th} row and s^{th} column of $\phi\psi$ is the sum of the products obtained by multiplying each constituent of the r^{th} row of ϕ into the corresponding constituent of the s^{th} column of ψ.

Since by definition the product of two matrices is associative with their operand, it follows that the multiplication of matrices is also associative. For by definition

$$\phi(\psi\chi)\cdot\rho = \phi\cdot(\psi\chi)\rho = \phi(\psi\cdot\chi\rho),$$

and likewise,

$$(\phi\psi)\chi\cdot\rho = (\phi\psi)\cdot\chi\rho = \phi(\psi\cdot\chi\rho).$$

As this is true for any quantity in the ground, hence

$$\phi(\psi\chi) = (\phi\psi)\chi.$$

The multiplication of matrices is also distributive over addition. For if the quantities ρ' and ρ'' denote severally the linear unit functions $\phi\rho$ and $\psi\rho$ of any quantity ρ in the ground, by §2 and §3,

$$\begin{aligned}\phi(\psi+\chi)\rho &= \phi(\psi\rho+\chi\rho) = \phi(\rho'+\rho'') \\ &= \phi\rho'+\phi\rho'' \quad = \phi\psi\rho+\phi\chi\rho \\ &= (\phi\psi+\phi\chi)\rho.\end{aligned}$$

The commutative principle does not in general hold in the calculus of matrices.

The converse of the product of two matrices is the product of their converses in the reverse order, as may readily be proved. This gives the formula

$$\breve{\phi\psi} = \breve{\psi}\breve{\phi}.$$

§6. Division is the operation inverse to multiplication. Since multiplication is not in general commutative, two signs are required for division. In the last section, two matrices being given, it was required to find their product in either order; the problem inverse to this is to find a matrix which when multiplied into or by a given matrix ϕ shall have as product a given matrix ψ. In general, this problem is susceptible of one or more solutions: the matrix or matrices which when multiplied into ϕ give ψ as product may be denoted by $(\psi:\phi)$; and those which when multiplied by ϕ give ψ as product may be denoted by $\dfrac{\psi}{\phi}$ or $\dfrac{|\psi}{\phi|}$. Division is therefore defined by the equations

$$(\psi:\phi)\phi = \psi,$$

$$\phi\left(\frac{\psi}{\phi}\right) = \psi.$$

Invertible multiplication is multiplication whose inverse is determinative. If the multiplication of matrices is invertible, it would follow that if $\chi\phi = \psi$, and $\chi'\phi = \psi$, then $\chi = \chi'$; likewise, if $\phi\chi = \psi$ and $\phi\chi' = \psi$, then $\chi = \chi'$. Hence division would be subject further to the conditions

$$(\psi\phi):\phi = \psi,$$
$$\frac{\phi\psi}{\phi} = \psi,$$

by means of which the above results may be obtained immediately. To assume that multiplication is invertible is equivalent to regarding the inverse operation as the direct operation performed with an inverse quantity or matrix, which shares the associative and other properties of matrices in general. This inverse matrix is termed the reciprocal.*

It may now be shown that in general there exists a reciprocal, and hence that, in general, multiplication is invertible. From the rule for the formation of the product of two matrices, it follows that, if from the array of constituents representing a matrix ϕ another matrix Φ be formed, in which each constituent of the first array is replaced by the logarithmic differential derivative with respect to that constituent of the determinant of the array (provided this determinant is not null), the product of ϕ and the converse of Φ in either order is equal to unity.† Hence $\breve{\Phi}$ may be denoted by ϕ^{-1} and is termed the *reciprocal* of ϕ. In other words, if $|\phi|$ denote the determinant of the array representing the matrix (which will in future be termed the *content* of the matrix), and if Δ_ϕ represent the differential operator

$$\left(\begin{array}{cccc} \dfrac{\partial}{\partial a_{11}} & \dfrac{\partial}{\partial a_{12}} & \cdots & \dfrac{\partial}{\partial a_{1\omega}} \\ \dfrac{\partial}{\partial a_{21}} & \dfrac{\partial}{\partial a_{22}} & \cdots & \dfrac{\partial}{\partial a_{2\omega}} \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial}{\partial a_{\omega 1}} & \dfrac{\partial}{\partial a_{\omega 2}} & \cdots & \dfrac{\partial}{\partial a_{\omega\omega}} \end{array}\right)$$

*The term invertible multiplication was employed by Mr. C. S. Peirce in his *Logic of Relatives*, Memoirs Am. Acad., Vol. IX, in which, and in Mr. Peirce's other writings, will be found the substance of this account of division. Mr. Peirce has shown that the only algebras in which division is always determinative are ordinary algebra, with and without the imaginary, and real (semi) quaternions, this Journal, Vol. IV.

† In the wording of this statement of the relation between ϕ and Φ I have followed Prof. Sylvester, this Journal, Vol. VI.

then
$$\phi^{-1} = \frac{\breve{\Delta}_\phi |\phi|}{|\phi|}.$$

A matrix whose content is zero is termed *vacuous*. The inverse processes are sometimes possible with such a matrix; but since a vacuous matrix has no reciprocal, the results are indeterminate. Thus if

$$\varpi = \begin{pmatrix} a & 0 \\ ma & 0 \end{pmatrix},$$
$$\chi = \begin{pmatrix} b & 0 \\ mb & 0 \end{pmatrix},$$

the matrix $\left(\frac{\chi}{\varpi}\right)$ which when multiplied by ϖ gives χ as product, is

$$\begin{pmatrix} \frac{b}{a} & 0 \\ c & d \end{pmatrix}$$

where c and d are any two scalars.

If $\phi\chi = 0$, it is evident that either both ϕ and ψ are vacuous, or one or both are zero.

From the definition of the reciprocal it follows that $(\breve{\phi})^{-1} = \widetilde{(\phi^{-1})}$. For taking the converse of both sides of the equation $\phi\phi^{-1} = 1$, we get $\widetilde{(\phi^{-1})}\breve{\phi} = 1$; hence $\widetilde{(\phi^{-1})} = (\breve{\phi})^{-1}$.

§7. Regarding a matrix as an operator, the problem inverse to that of finding the effect of a matrix ϕ upon any quantity ρ in the ground (i. e. of finding the product of ϕ into ρ) is, given a quantity σ in the ground, to find another quantity which ϕ will transform into σ. Only one sign is needed for the inverse of functional multiplication, which is defined by the equation

$$\phi\left(\frac{\sigma}{\phi}\right) = \sigma.$$

The quotient $\frac{\sigma}{\phi}$ is always determinate if ϕ is non-vacuous. If the matrix ϕ transforms any extension $(\alpha_1, \alpha_2, \ldots \alpha_m)$ into one of lower order $(\beta_1, \beta_2, \ldots \beta_n)$, and if $\sigma = y_1\beta_1 + y_2\beta_2 + \ldots + y_n\beta_n$, the problem to find a quantity which ϕ transforms into σ is in each case possible, but the quotient $\frac{\sigma}{\phi}$ is not in each case

determinate. If $\rho = x_1\alpha_1 + x_2\alpha_2 + \ldots . + x_m\alpha_m$, there does not exist in every case a quotient $\frac{\rho}{\phi}$.

Powers and Roots of a Matrix.

§8. If m is an integer, ϕ^m is of course that matrix which results from multiplying ϕ m times by itself. If n is also an integer, $\phi^{\frac{m}{n}}$ must be defined as that matrix whose n^{th} power is the m^{th} power of ϕ. The irrational scalar power ϕ^p of ϕ must be regarded, as in common algebra, as the limit of the series $\phi^{p'}$, $\phi^{p''}$, etc., where p', p'', etc., are successive approximations to p, i. e. if

$$\phi^{p'} = \begin{pmatrix} a'_{11} & a'_{12} & \text{etc.} \\ a'_{21} & a'_{22} & \text{etc.} \\ \text{etc.} & & \end{pmatrix}$$

and if the limits of the a''s as p' approaches the limit p are respectively the similarly situated constituents of the matrix

$$\psi = \begin{pmatrix} a_{11} & a_{12} & \text{etc.} \\ a_{21} & a_{22} & \text{etc.} \\ \text{etc.} & & \end{pmatrix},$$

then $\phi^p = \psi$. Whence $\phi^p - \phi^{p'}$ may be made as nearly equal to zero as we please by taking p' sufficiently near to p; and consequently, if $\phi^{p'}$ is susceptible of having a reciprocal, $\phi^p : \phi^{p'}$ may be made as near unity as we please by taking p' sufficiently near to p. It is obvious that if σ is the limit for any quantity ρ of $\phi^{p'}\rho$, as p' approaches the limit p, then $\phi^p\rho = \sigma$.

In general, the matrix ϕ^m where m is an integer has a reciprocal which may be denoted by $(\phi^m)^{-1}$. Prof. Sylvester shows, virtually as follows, that $(\phi^m)^{-1} = (\phi^{-1})^m$. Since $\phi\phi^{-1} = \phi^{-1}\phi = 1$, hence ϕ and ϕ^{-1} are commutative; consequently $1 = (\phi^{-1}\phi)^m = (\phi^{-1})^m\phi^m$, or $(\phi^m)^{-1} = (\phi^{-1})^m\phi^m (\phi^m)^{-1} = (\phi^{-1})^m$. Fractional and irrational powers of a matrix also have in general a reciprocal; and in this case also it may be shown that $(\phi^m)^{-1} = \phi^{-m}$. Whence it follows that for any two scalars $(\phi^m)^n = \phi^{(mn)}$.

It is obvious that $\phi^m\phi^n = \phi^{m+n}$ for any two integers m and n, being merely an expression of the associative principle; and it is sufficiently plain that with a proper understanding the equation holds for any two scalars.

Linear-form Representation of a Matrix.

§9. Let the set of forms or *vids**

$$\begin{matrix} (\alpha_1:\alpha_1) & (\alpha_1:\alpha_2) & (\alpha_1:\alpha_3) & \text{etc.,} \\ (\alpha_2:\alpha_1) & (\alpha_2:\alpha_2) & (\alpha_2:\alpha_3) & \text{etc.,} \\ (\alpha_3:\alpha_1) & (\alpha_3:\alpha_2) & (\alpha_3:\alpha_3) & \text{etc.,} \end{matrix}$$

as operators upon the quantities α_1, α_2, etc., be defined by the equations

$$(\alpha_r:\alpha_s)\,\alpha_s = \alpha_r,$$
$$(\alpha_r:\alpha_s)\,\alpha_t = 0;$$

and, when members of the set are operands, by

$$(\alpha_r:\alpha_s)\ (\alpha_s:\alpha_n) = (\alpha_r:\alpha_n),$$
$$(\alpha_r:\alpha_s)\ (\alpha_t:\alpha_n) = 0.$$

Then the expression

$$\begin{aligned} \Phi = a_{11}(\alpha_1:\alpha_1) &+ a_{12}(\alpha_1:\alpha_2) + a_{13}(\alpha_1:\alpha_3) + \text{etc.} \\ + a_{21}(\alpha_2:\alpha_1) &+ a_{22}(\alpha_2:\alpha_2) + a_{23}(\alpha_2:\alpha_3) + \text{etc.} \\ + a_{31}(\alpha_3:\alpha_1) &+ a_{32}(\alpha_3:\alpha_2) + a_{33}(\alpha_3:\alpha_3) + \text{etc.} + \text{etc.,} \end{aligned}$$

linear in these vids, considered as an operator upon any quantity

$$\rho = x_1\alpha_1 + x_2\alpha_2 + x_3\alpha_3 + \text{etc.,}$$

is identical with the matrix

$$\phi = \left(\begin{matrix} a_{11} & a_{12} & a_{13} & \text{etc.} \\ a_{21} & a_{22} & a_{23} & \text{etc.} \\ a_{31} & a_{32} & a_{33} & \text{etc.} \\ \text{etc.} & & & \end{matrix}\right)$$

Moreover, if Ψ is a similar expression in the vids $(\alpha_r:\alpha_s)$ in which their coefficients the a's have been replaced by b's with corresponding subscripts, then $m\Phi + n\Psi$, where m and n are scalars, regarded as an operator upon ρ, is clearly identical with the matrix $m\phi + n\psi$. Likewise $\Phi\Psi$ has the same effect upon any operand ρ as the matrix $\phi\psi$; for

* The term *vid* was introduced by Mr. C. S. Peirce to denote the *units* or *letters* of an algebra. It will be employed in what follows to denote the forms $(\alpha_r:\alpha_s)$, which will also be termed the elementary units of a matrix.

$$\Phi\Psi = \Sigma_r\Sigma_s a_{rs}(\alpha_r : \alpha_s) . \Sigma_r\Sigma_s b_{rs}(\alpha_r : \alpha_s)$$
$$= \Sigma_r\Sigma_s\Sigma_t a_{rs} b_{st}(\alpha_r : \alpha_t),$$

which corresponds to the matrix

$$\left(\begin{array}{llll} \Sigma_r a_{1r} b_{r1} & \Sigma_r a_{1r} b_{r2} & \Sigma_r a_{1r} b_{r3} & \text{etc.} \\ \Sigma_r a_{2r} b_{r1} & \Sigma_r a_{2r} b_{r2} & \Sigma_r a_{2r} b_{r3} & \text{etc.} \\ \text{etc.} & & & \end{array}\right|$$

Any complete system of ω^2 of these vids forms a pure algebra of a certain class termed by Clifford *quadrates;* and expressed in terms of these units is in what may be termed its canonical form.* I shall therefore call an algebra linear in ω^2 of these vids a quadrate algebra of order ω; and any expression linear in the vids, a *quadrate form.* The multiplication tables to which these algebras give rise are similar, and are immediately obtained from the laws to which the vids are subject. Thus if $\omega = 2$, let

$$\begin{array}{cc} i & j \\ k & l \end{array}$$

denote a complete set of four of these vids. These letters or units give an algebra whose multiplication table is

	i	j	k	l
i	i	j	0	0
j	0	0	i	j
k	k	l	0	0
l	0	0	k	l

This is the algebra (g_4) of Prof. Peirce's linear associative algebras, and is a form of quaternions. If $\omega = 3$, let

$$\begin{array}{ccc} i & j & k \\ l & m & n \\ p & q & r \end{array}$$

denote the complete set of vids of the quadrate algebra of order three; these

*It seems appropriate to have a term to express that form of an algebra in which its units are capable of a classification, according to the requirements of the analysis of Peirce's *Linear Associative Algebra.* For this purpose I employ the term canonical form. Thus the form of quaternions given below is its canonical form; Hamilton's units are expressions linear in those given below.

letters give the algebra which was termed *nonions* by Clifford; its multiplication table is

	i	j	k	l	m	n	p	q	r
i	i	j	k	0	0	0	0	0	0
j	0	0	0	i	j	k	0	0	0
k	0	0	0	0	0	0	i	j	k
l	l	m	n	0	0	0	0	0	0
m	0	0	0	l	m	n	0	0	0
n	0	0	0	0	0	0	l	m	n
p	p	q	r	0	0	0	0	0	0
q	0	0	0	p	q	r	0	0	0
	0	0	0	0	0	0	p	q	r

In virtue of the correspondence that has been shown in the first part of this section to exist between quadrate forms and matrices, it follows that to any function of one or more matrices corresponds a quadrate form which is the same function of one or more quadrate forms corresponding to the several matrices, and which has upon any operand ρ the same effect as the resulting matrix. Hence whatever equality subsists between combinations of matrices, if it can receive interpretation as like operations upon the ground, also subsists between the same combinations of corresponding quadrate forms. Thus it appears that there is no essential difference between the theory of matrices and the theory of quadrate forms. Viewed in this aspect, the scalar quantities arranged in a square and forming a matrix may be regarded as the scalar coefficients of the several vids. Or the substantial identity between the theory of matrices and of quadrate algebras may be brought out by considering all the possible ω^2 matrices of order ω which can be formed with one constituent unity and the remainder zero; when it is very readily seen that these matrices have the same multiplication table, and the same effect upon the ground as the ω^2 vids of the quadrate algebra of order ω; and since any matrix of order ω may be regarded as an expression in an algebra whose units are these ω^2 matrices, we thus have two algebras whose multiplication table is the same, and which, consequently, are identical. Hence a matrix is a quadrate form, and conversely. It is easily shown that the matrix of order ω whose non-zero constituent is in the r^{th} row and s^{th} column is identical

with the vid occupying the same place in the quadrate system of order ω. Thus if $\omega = 2$, the four matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

in terms of which any dual matrix may be expressed, are respectively identical with the units i, j, k, l of the quadrate algebra of order four, or their equals the vids $(\alpha_1 : \alpha_1)$, $(\alpha_1 : \alpha_2)$, $(\alpha_2 : \alpha_1)$, $(\alpha_2 : \alpha_2)$; the matrices have, moreover, the same effect upon the ground (α_1, α_2) as these vids, etc.

The discovery of these systems of vids or of quadrate algebras is due to Mr. C. S. Peirce, by whom, through this discovery, the relation of the theory of quadrate forms (or the theory of matrices) to the general theory of multiple algebra was first made clear.

By §3,
$$\begin{aligned} \breve{\phi} &= a_{11} \, \overset{\smile}{(\alpha_1 : \alpha_1)} + a_{12} \, \overset{\smile}{(\alpha_1 : \alpha_2)} + \text{etc.} \\ &\quad + a_{21} \, \overset{\smile}{(\alpha_2 : \alpha_1)} + a_{22} \, \overset{\smile}{(\alpha_2 : \alpha_2)} + \text{etc.} + \text{etc.} \end{aligned}$$

But by definition $\breve{\phi}$ is obtained from ϕ by interchanging its row and columns. Hence

$$\overset{\smile}{(\alpha_r : \alpha_r)} = (\alpha_r : \alpha_r), \quad \overset{\smile}{(\alpha_r : \alpha_s)} = (\alpha_s : \alpha_r).$$

The vids of the type $(\alpha_r : \alpha_r)$ I shall term self-transverse or self-converse vids; they may also be termed symmetric vids. Those of the type $(\alpha_r : \alpha_s)$ I shall term non-symmetric vids. Mr. Peirce terms the vids of the first type self-vids; the vids of the second type he terms alio-vids.

Sketch of the History of the Development of the Theory of Matrices.

§10. An outline of the origin of the theory of matrices was given in §1. Subsequent to Cayley's memoir, the next advance was made in 1870 by Charles S. Peirce, who, in his investigations upon the extension of the Boolian calculus to the logic of relatives,* came upon a set of forms (considered in §9) constituting a system virtually identical with the calculus of matrices. Peirce showed that any relative term involving not more than one correlate (dual relative)† could be represented as an expression linear in the units of a linear transformation.

* *Description of a Notation for the Logic of Relatives*, Memoirs Am. Acad. Sciences, Vol. IX (1870).
† Such as "lover of," "loved by," "mother of," etc.; but not "buyer of—from—," etc.

Whence follows the remarkable theorem that every such relation between any group of objects can be represented by a matrix. As has been stated, the relation of the theory of matrices, as algebras of a certain class (see §9), to linear associative algebra in general, was first made clear through the light thrown on the subject by Peirce's systems of vids.

Charles Peirce has made the great discovery that the whole theory of linear associative algebra is included in the theory of matrices. He has shown that every linear associative algebra has a relative form, i. e. its units may be expressed linearly in terms of the vids (denoted in his notation by $(A:A)$, $(A:B)$, etc.) of a linear transformation; and consequently, that any expression in the algebra can be represented by a matrix.* Whence the theory of all possible linear associative algebras is only the theory of all possible sets of matrices constituting a group in Benjamin Peirce's sense, i.e. which are such that the product of any two members of the set can be expressed linearly in terms of itself and the other members of the set alone. Charles Peirce has, moreover, given the relative or matrix form of all the algebras considered by his father in his *Linear Associative Algebra.*

To Charles Peirce, in conjunction with his father, the identification of quaternions with the quadrate algebra of order two (i. e. the algebra of dual matrices) is also due. Cayley, in his memoir, had remarked upon the similarity a certain system of three dual matrices had to the i, j, k of quaternion; but the identification was not completed until the remarkable discovery by Benjamin Peirce of a form of quaternions, which, in §9, I have termed the canonical form, and which results from choosing the linear functions

$$\frac{1+i\sqrt{-1}}{2},\quad \frac{j+k\sqrt{-1}}{2},\quad \frac{-j+k\sqrt{-1}}{2},\quad \frac{1-i\sqrt{-1}}{2},$$

of unity and any three mutually normal unit vectors i, j, k, as units of the algebra. These units obviously have the same multiplication table as the vids of a dual matrix. In his memoir of 1870 Charles Peirce had given, as an example of the infinite system of quadrate algebras, the multiplication table of the quadrate algebra next in order after quaternions, afterwards named *nonions* by Clifford; and he states in the Johns Hopkins Univ. Circs. No. 22 (April,

* See Mr. Peirce's *Logic of Relatives*, above referred to; also this Journal, Vol. IV, p. 221, and the Proc. Am. Acad. of Arts and Sciences for 1875.

1882), that the identification of quaternions with the quadrate algebra of order two suggested to his father and himself that, as nonions was thus shown to be the exact analogue of quaternions, there ought to be a form of nonions analogous to Hamilton's tetranomial form of quaternions, which form of nonions, Mr. Peirce states, either his father or himself found.* By means of this form any expression in the algebra is susceptible of representation linearly in terms of unity and eight non-scalar cube roots of unity, just as any expression in quaternions is susceptible of representation in terms of unity and three square roots of unity (or of -1). This form of nonions I shall term the octanomial form. The Peirces' discovery of the octanomial form of nonions was not published. The priority of publication of this form belongs to Sylvester, who discovered it subsequently to the Peirces without any knowledge of their investigations upon nonions.†

To Sylvester we owe most of the development of the theory of matrices. In his unfinished memoir in this Journal,‡ Sylvester distinguished between the different degree of vacuity and nullity of a matrix, replacing by these terms the term indeterminate used rather vaguely by Cayley to denote a null or vacuous matrix. It should be stated that Clifford had previously distinguished between the different degrees of nullity, employing the term indeterminate, with the prefix singly, doubly, etc. In this memoir Sylvester showed also how to derive the *chain of equations* from the identical equation, and the relation of these to the latent function of two or more matrices taken in a certain order. In a series of papers in the Johns Hopkins Univ. Circulars and in the Phil. Mag., Sylvester added largely to the theory of matrices.‖ He demonstrated the extension of Hamilton's theorem concerning the cubic equation to which every matrix of the third order is subject, which was enunciated without proof by Cayley in 1858; among the other important results are what I term the second branch of the law of nullity, the corollary of the law of nullity, the law of latency, and the expression for any function of a single matrix. Sylvester has also extended the solution of

* "So much was published by me in 1870 [the multiplication table of the canonical form of nonions, etc.], and it then occurred to my father or to me (probably in conversing together) that since this algebra was thus shown (through his form of quaternions) to be the strict analogue of quaternions, there ought to exist a form of it analogous to Hamilton's standard tetranomial form. That form either he or I certainly found. I cannot remember which after so many years; whichever did must have found it at once."—*Johns Hopkins Univ. Circs.* No. 22 (April, 1882).

† Johns Hopkins Univ. Circs. Nos. 15 and 17 (1882).

‡ "Lectures on Universal Multiple Algebra," this Journal, Vol. VI (1883).

‖ Nos. 15, 17, 27, 28, 32.

equations in quaternions or dual matrices beyond the point where Hamilton left the subject.

Clifford's contributions to the theory of matrices have been left to the last, though in point of time they preceded Sylvester's researches. They are contained in his "Fragment on Matrices," published in the posthumous edition of his works. I shall show subsequently that the main proposition of which he gives a proof is false. But the basis of his treatment of the subject is an important contribution to the theory of matrices. It is the method which is adopted in this paper; and the demonstration which I have given in the second part of this paper of the law of nullity is based on a result contained in Clifford's "Fragment."

II.—Development of the Theory by Means of the Axes of a Matrix.

Axes of a Matrix, or Quantities for which $(\phi - g)\rho = 0$, where g is a Scalar.

§1. If $\rho = \Sigma x_1 \alpha_1$ and $(\phi - g)\rho = 0$, then

$$
\begin{aligned}
&(a_{11} - g)x_1 + a_{12}x_2 + \dots + a_{1\omega}x_\omega = 0, \\
&a_{21}x_1 + (a_{22} - g)x_2 + \dots + a_{2\omega}x_\omega = 0, \\
&\text{etc., etc.}
\end{aligned}
$$

The resultant of this system is called the *latent function* of ϕ, and will be denoted by $|\phi - g|$, as it is the determinant of the matrix $\phi - g$. From its vanishing results an equation of order ω in g,

$$g^\omega - m_{\omega-1}g^{\omega-1} + m_{\omega-2}g^{\omega-2} - \dots \mp m_1 g \pm m = 0.$$

The constant term m is evidently the determinant of the array of constituents forming the matrix, and has been denoted by $|\phi|$; m_1 is the sum of all the principal first minors of $|\phi|$; and, generally, $m_{\omega-\kappa}$ is the sum of all the principal κ^{th} minors of $|\phi|$. The term latent function is due to Prof. Sylvester, and the roots $g_1, g_2, \dots g_\omega$ of the ω^{ic} he terms the *latent roots* of ϕ. If $|\phi| = 0$, it has already been stated that ϕ is then termed vacuous, and evidently has one latent root zero. If, in addition, $m_1 \neq 0$, ϕ has only one latent root zero, and is then said to have the vacuity one, or is simply vacuous. More generally, if all the m's from m to $m_{\kappa-1}$ are zero, and $m_\kappa \neq 0$, the matrix ϕ has just κ latent roots zero, and is said to have the vacuity κ. If $|\phi| \neq 0$, ϕ is non-vacuous, or its vacuity is zero.

Corresponding respectively to each latent root of ϕ are the ω quantities $\rho_1, \rho_2, \ldots . \rho_\omega$, such that

$$0 = (\phi - g_1)\rho_1 = (\phi - g_2)\rho_2 = \ldots . = (\phi - g_\omega)\rho_\omega .$$

These quantitities will be termed the *axes of* ϕ.

§2. There is no linear relation between any set of the axes of ϕ corresponding to distinct latent roots. For suppose any set of n axes are linearly related which correspond to the distinct latent roots $g_1, g_2, \ldots . g_n$. Let κ be the number of these axes which are linearly independent, and suppose they correspond to the first κ latent roots. Then any axis corresponding to any of the remaining $n - \kappa$ latent roots can be expressed linearly in terms of these. Thus

$$\rho_{\kappa+1} = t_1\rho_1 + t_2\rho_2 + \ldots . + t_\kappa\rho_\kappa,$$

$$\therefore\ g_{\kappa+1}\rho_{\kappa+1} = \phi\rho_{\kappa+1} = t_1 g_1\rho_1 + t_2 g_2\rho_2 + \ldots . + t_\kappa g_\kappa\rho_\kappa;$$

but, since the t's are not all zero, this is impossible.

Whence, *if all the latent roots are distinct, the axes of ϕ are all linearly independent.* If a set of latent roots become equal, linear relations may arise between the set of axes corresponding to them, i. e. certain of these axes may be projected into the extension of the remaining axes corresponding to that set of latent roots, or all the axes of the set may become coincident.

If two or more axes of the set remain linearly independent when the set of latent roots become equal, these axes and also the remaining axes become indeterminate. Thus if the n latent roots $g_1, g_2, \ldots . g_n$ ultimately become equal, of the axes corresponding to them only the first κ may remain linearly independent, and the remaining $n - \kappa$ will then be expressible linearly in terms of them. These κ axes $\rho_1, \rho_2, \ldots . \rho_\kappa$ will all satisfy the equation $(\phi - g_1)\rho = 0$, and consequently any expression $x_1\rho_1 + x_2\rho_2 + \ldots . + x_\kappa\rho_\kappa$ linear in them will also satisfy this equation, and hence will be an axis of ϕ. In this case any κ quantities giving the extension of $\rho_1, \rho_2, \ldots . \rho_\kappa$, together with any $n - \kappa$ other quantities in their extension, may be regarded as the axes of ϕ corresponding to the n-fold latent root g_1. No quantity in the extension of $\rho_1, \rho_2, \ldots . \rho_\kappa$ can be in any linear relation with axes corresponding to latent roots other than g_1 by what has just been proved, as such quantities are axes of ϕ corresponding to g_1.

The matrix $\psi = \phi - g_1$ evidently has as latent roots $g_1 - g_1, g_2 - g_1, g_3 - g_1$, etc.; for if g is any latent root of ϕ, then $|\psi - (g - g_1)| = |(\phi - g_1) - (g - g_1)| = |\phi - g| = 0$; and hence $g - g_1$ is a latent root of ψ, since it is a root of the

latent function of ψ. Conversely, if $g - g_1$ is a latent root of ψ, g is a latent root of ϕ. Now if the axis of ϕ corresponding to g_1 (i. e. the axes of ψ corresponding to zero) is indeterminate, then every first minor of $|\phi - g_1| = |\psi|$ is zero; consequently every principal first minor of $|\psi|$ is zero; but then ψ has two latent roots zero, and consequently ϕ has two latent roots equal to g_1. Hence the axes corresponding to latent roots occurring only once among the latent roots of ϕ are never indeterminate.

Identical Equation.

§3. If ϕ is a matrix all of whose latent roots are distinct, any quantity ρ in the extension of the ground may be represented linearly in terms of its axes. Thus for any quantity ρ we may put

$$\rho = z_1\rho_1 + z_2\rho_2 + \dots + z_\omega\rho_\omega .$$

Operating upon ρ by $\phi - g_\omega$ we get

$$(\phi - g_\omega)\rho = z_1(g_1 - g_\omega)\rho_1 + z_2(g_2 - g_\omega)\rho_2 + \dots + z_\omega(g_\omega - g_\omega)\rho_\omega .$$

By this operation the last component of ρ, that along ρ_ω, is annulled. Operating on $(\phi - g_\omega)\rho$ by $\phi - g_{\omega-1}$, another component of ρ is annulled. Finally

$$(\phi - g_1)(\phi - g_2)\dots(\phi - g_{\omega-1})(\phi - g_\omega)\rho = 0 .$$

This being true for any quantity ρ in the ground, may be written

$$(\phi - g_1)(\phi - g_2)\dots(\phi - g_{\omega-1})(\phi - g_\omega) \equiv \phi^\omega - m_{\omega-1}\phi^{\omega-1} + \dots \mp m_1\phi \pm m = 0 .$$

This is Cayley's "identical equation."

The proof of the identical equation may be extended to any case as follows: The latent roots being supposed to be all distinct, let $h_2 = g_2 - g_1$, $h_3 = g_3 - g_1$, etc., $h_n = g_n - g_1$, and let $\phi - g_1$ be denoted by ψ; finally, let χ denote the product $(\phi - g_{n+1})(\phi - g_{n+2})\dots(\phi - g_\omega)$,—the identical equation then becomes

$$[\psi^n - \Sigma h_1\psi^{n-1} + \text{etc.}]\,\chi = \psi^n\chi - \Sigma h_1\psi^{n-1}\chi + \text{etc.} = 0 .$$

This equation is true however small the h's may be, provided the latent roots remain distinct. Consequently, as the h's diminish without limit, the first term of the second member $\psi^n\chi$ must ultimately diminish without limit, and in the limit the identical equation becomes

$$\psi^n\chi = (\phi - g_1)^n(\phi - g_{n+1})\dots(\phi - g_\omega) = 0 .$$

Thus it is proved that the identical equation still subsists when any group of n latent roots becomes equal, the other latent roots being distinct. In the same way it may be shown that if any other set of the latent roots become equal, and finally whatever the relation between the latent roots, the identical equation subsists.

Cayley has remarked that the identical equation may be represented as follows:

$$\begin{vmatrix} a_{11}-\phi & a_{12} & \dots\dots & a_{1\omega} \\ a_{21} & a_{22}-\phi & \dots\dots & a_{2\omega} \\ \dots & \dots & \dots & \dots \\ a_{\omega 1} & a_{\omega 2} & \dots\dots & a_{\omega\omega}-\phi \end{vmatrix} = 0 .$$

Or, as stated by Cayley, the determinant of the matrix less the matrix itself considered as involving the matrix unity is zero. This relation Cayley denotes symbolically by Det $(1\phi - \phi\tilde{1}) = 0$, where $\phi\tilde{1}$ signifies that ϕ is to be treated as a scalar. I propose to denote ϕ considered as involving the matrix unity by $\tilde{\phi}$, when, with the notation previously employed for the determinant of a matrix, the identical equation may be represented by $|\phi - \tilde{\phi}| = 0$.

Corresponding to any latent root g_1 of ϕ can, by §1, always be found an axis of ϕ; and upon this the effect of any operator $\phi - g_2$ for a latent root g_2 distinct from g_1 is by §2 only to multiply it by a scalar constant. Consequently no product of factors $\phi - g$, not containing $\phi - g_1$, can annul the axis ρ_1, and hence no such product can vanish. Similarly for every other latent root. Whence the identical equation must contain a factor $\phi - g$ for each distinct latent root. On the other hand, it was shown in §2 that when the latent roots are not all distinct, there may be more than one axis corresponding to a latent root g_1 which occurs more than once. In this case $\phi - g_1$ will annul an extension of order greater than unity, and the order of the identical equation will be lowered.

It is obvious that there can be but one identical equation, and that if ϕ is subject to any other equation involving only the matrix ϕ, this must contain the equation $|\phi - \tilde{\phi}| = 0$ as a factor.

Converse of ϕ.

§4. The latent function $|\breve{\phi} - g|$ of $\breve{\phi}$ is obviously identical with the latent function of ϕ. Consequently the latent roots of ϕ and $\breve{\phi}$ are identical, and the identical equation in $\breve{\phi}$ is

$$(\breve{\phi} - g_1)(\breve{\phi} - g_2) \dots\dots (\breve{\phi} - g_\omega) = 0 .$$

The axes of ϕ will be denoted by $\breve{\rho}_1, \breve{\rho}_2, \ldots \breve{\rho}_\omega$. They do not necessarily coincide with those of ϕ. Indeed, I will show later that the axis $\breve{\rho}_1$ of $\breve{\phi}$ corresponding respectively to the latent root g_1 is normal to all the axes of ϕ, $\rho_2, \rho_3, \ldots \rho_\omega$ corresponding to the remaining latent roots.

On the Form Representing a Matrix.

§5. If the latent roots of ϕ are assigned, ϕ is subject to ω conditions. If in addition ω linearly independent axes are assigned to which the assigned latent roots are to correspond, ϕ is by definition completely determined. When the latent roots are given as not all distinct, and corresponding to them ω axes, of which those corresponding to a set of equal latent roots are not in every case linearly independent, ϕ is not completely determined; an infinity of matrices can be found which will have these latent root and corresponding axes, and the same identical equation.

To find the *form* or *array* representing the matrix ϕ whose latent roots g_1, g_2, etc., and corresponding axes ρ_1, ρ_2, etc., are assigned, we may proceed as follows. Let

$$\begin{aligned}\rho_1 &= x_{11}\alpha_1 + x_{12}\alpha_2 + \ldots + x_{1\omega}\alpha_\omega,\\ \rho_2 &= x_{21}\alpha_1 + x_{22}\alpha_2 + \ldots + x_{2\omega}\alpha_\omega,\\ &\ldots\ldots\ldots\ldots\ldots\ldots\\ \rho_\omega &= x_{\omega 1}\alpha_1 + x_{\omega 2}\alpha_2 + \ldots + x_{\omega\omega}\alpha_\omega.\end{aligned}$$

Denoting by X the matrix formed from the array of the x's, evidently

$$\rho_1 = \breve{X}\alpha_1, \quad \rho_2 = \breve{X}\alpha_2, \quad \rho_3 = \breve{X}\alpha_3, \text{ etc.},$$

and by definition

$$\phi(\breve{X}\alpha_1) = g_1\breve{X}\alpha_1, \quad \phi(\breve{X}\alpha_2) = g_2\breve{X}\alpha_2, \quad \phi(\breve{X}\alpha_3) = g_3\breve{X}\alpha_3, \text{ etc.},$$

i. e.
$$(\phi \between \breve{X} \between \alpha_1, \alpha_2, \ldots \alpha_\omega) = (\breve{X} \between \begin{vmatrix} g_1 & 0 & \ldots & 0 \\ 0 & g_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & g_\omega \end{vmatrix} \between \alpha_1, \alpha_2, \ldots \alpha_\omega)$$

As this equation is true for each of the α's, it is true for any expression linear in them, and so for any quantity in the ground. Hence

$$\phi = (\breve{X} \between \begin{vmatrix} g_1 & 0 & \ldots & 0 \\ 0 & g_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & g_\omega \end{vmatrix} \between \breve{X}^{-1})$$

This method is substantially identical with that given in Clifford's "Fragment on Matrices" by Cayley as a correction of Clifford's result.

If a set of ω linearly independent quantities other than the α's are chosen to represent the ground, the linear unit operator or matrix ϕ will, in general, be represented by another form or array. Two exceptions are, however, to be noted, the matrix unity and the matrix zero always have the same representation. By §2, if the latent roots of ϕ are all distinct, the axes of ϕ are capable of representing the ground. If they are chosen for this purpose, the form representing ϕ becomes

$$\begin{pmatrix} g_1 & 0 & \dots & 0 \\ 0 & g_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & g_\omega \end{pmatrix}$$

Evidently this form of a matrix gives an immediate proof of the identical equation. It has just been shown how from this form to pass to that form of ϕ when the α's are chosen to represent the ground, the axes being given in terms of the α's. If the α's are given in terms of the axes, and $\alpha_1 = \breve{X}\rho_1$, $\alpha_2 = \breve{X}\rho_2$, etc., the only change in the formula for ϕ is that $\breve{X}$ is to be replaced by $\breve{X}^{-1}$.

The more general problem, given the form representing a matrix ϕ in terms of certain ω linearly independent quantities (i. e. the form of ϕ, when these quantities are chosen to represent the ground), to find the form of ϕ in terms of another set of ω linearly independent quantities is solved in a similar way. Denote by ϕ_α the form of ϕ in terms of the set $(\alpha_1, \alpha_2, \dots \alpha_\omega)$, and by ϕ_β the form of ϕ in terms of the set $(\beta_1, \beta_2, \dots \beta_\omega)$, and let

$$\phi_\alpha = \begin{pmatrix} a_{11} & a_{12} & \text{etc.} \\ a_{21} & a_{22} & \text{etc.} \\ \text{etc.} & & \end{pmatrix} \qquad \phi_\beta = \begin{pmatrix} b_{11} & b_{12} & \text{etc.} \\ b_{21} & b_{22} & \text{etc.} \\ \text{etc.} & & \end{pmatrix}$$

If

$$\begin{aligned} \beta_1 &= x_{11}\alpha_1 + x_{12}\alpha_2 + \dots + x_{1\omega}\alpha_\omega, \\ \beta_2 &= x_{21}\alpha_1 + x_{22}\alpha_2 + \dots + x_{2\omega}\alpha_\omega, \\ &\dots\dots\dots\dots \\ \beta_\omega &= x_{\omega 1}\alpha_1 + x_{\omega 2}\alpha_2 + \dots + x_{\omega\omega}\alpha_\omega, \end{aligned}$$

and X represents the matrix formed from the array of x's; then

$$\beta_1 = \breve{X}\alpha_1, \quad \beta_2 = \breve{X}\alpha_2, \quad \beta_3 = \breve{X}\alpha_3, \quad \text{etc.}$$

Operating by the matrix ϕ upon the β's and their equivalents gives

$$\phi_\alpha \breve{X}\alpha_1 = \phi_\beta \beta_1, \quad \phi_\alpha \breve{X}\alpha_2 = \phi_\beta \beta_2, \quad \phi_\alpha \breve{X}\alpha_3 = \phi_\beta \beta_3, \quad \text{etc.}$$

For, by definition, the effect of ϕ_α upon any expression in the α's is identical with the effect of ϕ_β upon the equivalent expression in the β's, since ϕ_α and ϕ_β both represent the same matrix. It is not proper yet to replace the β's by their expressions in terms of the α's, for ϕ_β as an operator upon the α's is not equivalent to the effect of the matrix ϕ. As the result of operating upon the β's by ϕ_β we have

$$\begin{aligned}
\phi_\alpha \breve{X}\alpha_1 &= b_{11}\beta_1 + b_{21}\beta_2 + \dots\dots + b_{\omega 1}\beta_\omega, \\
\phi_\alpha \breve{X}\alpha_2 &= b_{12}\beta_1 + b_{22}\beta_2 + \dots\dots + b_{\omega 2}\beta_\omega, \\
&\dots\dots\dots\dots\dots\dots\dots\dots \\
\phi_\alpha \breve{X}\alpha_\omega &= b_{1\omega}\beta_1 + b_{2\omega}\beta_2 + \dots\dots + b_{\omega\omega}\beta_\omega.
\end{aligned}$$

Replacing now the β's by their equivalents,

$$\begin{aligned}
\phi_\alpha \breve{X}\alpha_1 &= b_{11}\breve{X}\alpha_1 + b_{21}\breve{X}\alpha_2 + \dots\dots + b_{\omega 1}\breve{X}\alpha_\omega = \breve{X}\phi_\beta\alpha_1, \\
\phi_\alpha \breve{X}\alpha_2 &= b_{12}\breve{X}\alpha_1 + b_{22}\breve{X}\alpha_2 + \dots\dots + b_{\omega 2}\breve{X}\alpha_\omega = \breve{X}\phi_\beta\alpha_2, \\
&\text{etc.} \qquad \text{etc.}
\end{aligned}$$

i. e.
$$\phi_\alpha \breve{X}(\alpha_1, \alpha_2, \dots\dots \alpha_\omega) = \breve{X}\phi_\beta(\alpha_1, \alpha_2, \dots\dots \alpha_\omega),$$

where ϕ_β is now regarded as an operator upon the α's,

$$\therefore\ \phi_\alpha = \breve{X}\phi_\beta\breve{X}^{-1}.$$

§6. From the definition of a matrix as an operator upon a certain ground it follows that the identical equation does not vary with the form. It may, however, formally be proved as follows: Since $\phi_\alpha = \breve{X}\phi_\beta\breve{X}^{-1}$, evidently $\phi_\alpha^2 = \breve{X}\phi_\beta^2\breve{X}^{-1}$, etc., let the identical equation in ϕ_α be

$$\begin{aligned}
\phi_\alpha^\omega - m_{\omega-1}\phi_\alpha^{\omega-1} + \dots\dots \mp m_1\phi_\alpha \pm m \qquad &= 0, \\
\therefore\ \breve{X}(\phi_\beta^\omega - m_{\omega-1}\phi_\beta^{\omega-1} + \dots\dots \mp m_1\phi_\beta \pm m)\breve{X}^{-1} &= 0.
\end{aligned}$$

But since the β's constitute a set of ω linearly independent quantities, $|\breve{X}| \neq 0$. Consequently,

$$\phi_\beta^\omega - m_{\omega-1}\phi_\beta^{\omega-1} + \dots\dots \mp m_1\phi_\beta \pm m = 0.$$

Whence the vacuity of a matrix does not vary with its form. This proof of the invariance of the vacuity does not, however, take into consideration the fact that the identical equation may degrade. The invariance may, however, be shown as follows: If g is a scalar $\breve{X}g\breve{X}^{-1} = g$; hence

$$\phi_\alpha - g = \breve{X}\phi_\beta\breve{X}^{-1} - \breve{X}g\breve{X}^{-1} = \breve{X}(\phi_\beta - g)\breve{X}^{-1}.$$

From the definition of the product of two matrices it follows that the determinant of their product is equal to the product of their determinants.

$$\therefore\ |\phi_\alpha - g| = |\breve{X}|\,|\phi_\beta - g|\,|\breve{X}^{-1}| = |\phi_\beta - g|;$$

for $|\breve{X}^{-1}| = |\breve{X}|^{-1}$. Hence not only are the distinct latent roots of ϕ unchanged with its form, but the number of times each latent root is repeated is also unchanged.

Definition of Nullity and the Law of Nullity.

§7. A null matrix is one whose determinant vanishes, or of which all the minors of a certain order vanish. A non-null matrix is said to have a nullity zero, and one, every constituent of which is zero, is said to be absolutely null, or to have the nullity ω. It has been shown that the absolutely null matrix is the scalar quantity zero. Between these limits the number expressing the measure of nullity may have any integer value. If all the $(\varkappa - 1)^{\text{th}}$ minors of the determinant of a matrix vanish, but not all the $\varkappa^{\text{th}}$ minors, the matrix has a nullity $\varkappa$. Nullity of order one or simple nullity is evidently the same as simple vacuity. The vacuity of a matrix obviously cannot exceed its nullity, but it may have simple nullity and vacuity of any order from unity to ω.

The nullity of ϕ is not affected by multiplying it by a non-null matrix. Thus if the nullity of ϕ is m, the nullity of ϖ zero, the nullity of $\phi\varpi = \psi$ is m. For the $(m-1)^{\text{th}}$ minors of ψ consist of all possible products of a rectangular determinant formed from $\omega - m + 1$ rows of ϕ into the rectangular determinant formed from the corresponding $\omega - m + 1$ columns of ϖ; and each of these products is resolvable into the sum of $\omega - m + 1$ products of an $(m-1)^{\text{th}}$ minor of ϕ into an $(m-1)^{\text{th}}$ minor of ϖ. But the $(m-1)^{\text{th}}$ minors of ϕ are all zero. Whence the nullity of ψ is not less than m.

Since, however, $|\varpi| \neq 0$, $\phi = \psi\varpi^{-1}$, and in the same way it may be shown that the nullity of ϕ is not less than the nullity of ψ; hence the nullity of ψ is m. By the same method it may be shown that if $\varpi\phi = \psi$, the nullity of ψ is still equal to that of ϕ.* It follows immediately from this that the nullity of a matrix is unchanged when the form representing it is changed.

* This proof follows the method employed by Sylvester to prove the theorem regarding the lower limit of the nullity of the product of two matrices. To Sylvester, as has been stated (I, §10), the term nullity and the discrimination between degrees of nullity are due.

§8. *If the greatest extension annulled by any matrix is m, its nullity is m, and conversely.* This I call the *first branch of the law of nullity.* Let $\alpha_1, \alpha_2, \ldots \alpha_m$ be any m linearly independent quantities in the greatest extension A annulled by ϕ; and let $\beta_1, \beta_2, \ldots \beta_n$ be any $n = \omega - m$ quantities in the extension B complementary to A with respect to the ground; the α's and β's may then be chosen to represent the ground, and any quantity ρ in the ground may be expressed by

$$\rho = x_1\alpha_1 + x_2\alpha_2 + \ldots + x_m\alpha_m + y_1\beta_1 + y_2\beta_2 + \ldots + y_n\beta_n,$$
$$\therefore\ \phi\rho = y_1\beta_1 + y_2\phi\beta_2 + \ldots + y_n\phi\beta_n.$$

If
$$\begin{aligned}
\phi\beta_1 &= a_{11}\alpha_1 + a_{21}\alpha_2 + \ldots + a_{m1}\alpha_m + b_{11}\beta_1 + b_{21}\beta_2 + \ldots + b_{n1}\beta_n,\\
\phi\beta_2 &= a_{12}\alpha_1 + a_{22}\alpha_2 + \ldots + a_{m2}\alpha_m + b_{12}\beta_1 + b_{22}\beta_2 + \ldots + b_{n2}\beta_n,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
\phi\beta_n &= a_{1n}\alpha_1 + a_{2n}\alpha_2 + \ldots + a_{mn}\alpha_m + b_{1n}\beta_1 + b_{2n}\beta_2 + \ldots + b_{nn}\beta_n,
\end{aligned}$$

then ϕ is represented by the form

$$\left[\begin{array}{c|cccc}
 & a_{11} & a_{12} & \ldots & a_{1n}\\
 & a_{21} & a_{22} & \ldots & a_{2n}\\
 & \cdots & \cdots & \cdots & \cdots\\
 & a_{m1} & a_{m2} & \ldots & a_{mn}\\
\hline
 & b_{11} & b_{12} & \ldots & b_{1n}\\
 & b_{21} & b_{22} & \ldots & b_{2n}\\
 & \cdots & \cdots & \cdots & \cdots\\
 & \cdots & \cdots & \cdots & \cdots\\
 & b_{n1} & b_{n2} & \ldots & b_{nn}
\end{array}\right]$$

where the blank square and rectangle consist of m columns of zeros. Hence ϕ has a nullity at least m. If the nullity exceeds m, the determinant Δ formed by the n columns of the b's must vanish, and likewise every determinant which can be formed from $n = \omega - m$ rows of the last n columns of ϕ. Denoting the first minors of Δ, corresponding respectively to b_{11}, b_{12}, etc., by B_{11}, B_{12}, etc., we shall then have

$$\phi\,(B_{11}\beta_1 + B_{12}\beta_2 + \ldots + B_{1n}\beta_n) = B_{11}\phi\beta_1 + B_{12}\phi\beta_2 + \ldots + B_{1n}\phi\beta_n = 0.$$

But by supposition the extension A contains all the axes of ϕ corresponding to

the latent root zero.* Hence the nullity of ϕ cannot exceed m. The converse is immediately evident.

§9. *If the matrix ϕ transforms every quantity in the ground into one in an extension of order $\omega - m$, the nullity of ϕ is m, and conversely.* For if

$$\rho = x_1\alpha_1 + x_2\alpha_2 + \ldots + x_\omega\alpha_\omega$$

is any quantity in the ground, and

$$\phi\rho = (a_{11}x_1 + a_{12}x_2 + \ldots + a_{1\omega}x_\omega)\,\alpha_1 + (a_{21}x_1 + a_{22}x_2 + \ldots + a_{2\omega}x_\omega)\,\alpha_2 + \ldots + (a_{\omega 1}x_1 + a_{\omega 2}x_2 + \ldots + a_{1\omega}x_\omega)\,\alpha_\omega$$

is contained in an extension of order $\omega - m$, then there are m linear relations between the coefficients of the α's in the last equation true for all values of the x's. These give m sets of ω equations between the constituents of the matrix; namely,

$$\begin{aligned} l_1a_{11} + l_2a_{21} + \ldots + l_\omega a_{\omega 1} &= 0, \\ l_1a_{12} + l_2a_{22} + \ldots + l_\omega a_{\omega 2} &= 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ l_1a_\omega + l_2a_{2\omega} + \ldots + l_\omega a_{\omega\omega} &= 0, \end{aligned}$$

and $m - 1$ other sets of ω equations, in which the l's are replaced in each set by other constants. From the m equations obtained by taking the first equation in each set, $m - 1$ of the a's may be eliminated, leaving one equation between any $\omega - m + 1$ of the constituents of the first column; and, similarly, an equation linear in the corresponding constituents of the second column may be obtained from the second equations of the m sets, etc. Thus result ω equations linear in the constituents of each column taken from any $\omega - m + 1$ rows of $|\phi|$, the coefficients of the a's being the same for all constituents in the same row. The resultants of all sets of $\omega - m + 1$ of these equations, together with the like resultants formed from all such groups of ω equations, constitute the $(m - 1)^{\text{th}}$ minors of $|\phi|$; and these resultants must be zero. Whence the proposition. The converse is evident.

If, as in the preceding section, A denote the total extension annulled by ϕ, and B the complementary extension, it will not in general happen that B is the

* If not only Δ but also all the B's vanish, the analysis needs correction. But, since the constituents in the last n columns of ϕ do not all vanish, the minors of order p (for some value p from $p = 1$ to $p = n - 1$) do not all vanish; and then in a similar way, if the minors of order greater than p all vanish, it may be shown that there would result a linear relation between $\phi\beta_1$, $\phi\beta_2$, etc., i. e. ϕ would annul a certain expression linear in the β's.

extension into which ϕ transforms any quantity of the ground. As will appear later, this is only true when the vacuity of ϕ does not exceed its nullity. Let B_1 denote that part of B transformed by ϕ into A. There must exist such an extension unless no extension but A is annulled by any power of ϕ. For if $\phi^{\kappa+1}$ (where $\kappa \neq 0$) is the lowest power of ϕ that annuls an extension C other than A, then $\phi^{\kappa} C$ is included in A, since by supposition A is the total extension annulled by ϕ; moreover, $\phi^{\kappa-1} C$ is not included in A, since it is not annulled by ϕ; hence $\phi^{\kappa-1} C$ is a part of the ground complementary to A that is transformed by ϕ into A. Let B_2 denote that part of the extension $B - B_1$ which is transformed by ϕ into B_1. By an argument similar to that above, it may be shown that if the order of B_2 is zero, then no power of ϕ annuls an extension other than the aggregate of A and B_1, i. e. $A + B_1$; etc. Finally, let B_p denote that part of the extension $B - (B_1 + B_2 + \ldots + B_{p-1})$ which is transformed by ϕ into B_{p-1}; and suppose no portion of the remaining extension

$$B_{p+1} = B - (B_1 + B_2 + \ldots + B_{p-1} + B_p)$$

is transformed by ϕ into B_p. Then all the extensions A, B_1, B_2, ... B_{p-1}, B_p are annulled by some power of ϕ, but no part of the extension B_{p+1} complementary to their aggregate is annulled by any power of ϕ. The following multiplication table shows the effect of ϕ upon these mutually exclusive extensions:

	A	B_1	B_2	etc.	B_{p-1}	B_p	B_{p+1}
ϕ	0	A'	B_1'	....	B_{p-2}'	B_{p-1}'	B_{p+1}'
ϕ^2	0	0	A''	....	B_{p-3}''	B_{p-2}''	B_{p+1}''
ϕ^3	0	0	0	....	B_{p-4}'''	B_{p-3}'''	B_{p+1}'''
etc.	etc.	....	....	....	....	....	....
ϕ^{p-1}	0	0	0		$A^{(p-1)}$	$B_1^{(p-1)}$	$B_{p+1}^{(p-1)}$
ϕ^p	0	0	0		0	$A^{(p)}$	$B_{p+1}^{(p)}$

In this table the accented letters denote extensions included each in the extensions denoted by the same letter with fewer accents or unaccented.

The total extension A of order m annulled by ϕ I shall term the null extension or *null region* of the matrix, and the aggregate of the extensions exclusive of A annulled by some power of ϕ the vacuous extension or *vacuous region* of the matrix. The extension or region into which ϕ project any quantity of the ground is

$$\phi\,(A + B_1 + B_2 + \ldots\ldots + B_{p-1} + B_p + B_{p+1}) \\ = A' + B'_1 + B'_2 + \ldots\ldots + B'_{p-1} + B'_{p+1};$$

and this I shall term the *residual region* of ϕ. If $\{A'\}$ denote the order of the extension A', etc., evidently

$$\{A'\} \leqq \{B_1\},\quad \{B'_1\} \leqq \{B_2\}\ \text{ etc. }\ \{B'_{p-1}\} \leqq \{B_p\},\quad \{B'_{p+1}\} \leqq \{B_{p+1}\},^*$$

$$\therefore\ \{A'\} + \sum_1^{p-1}{}_{\varkappa}\, \{B'_\kappa\} + \{B'_{p+1}\} \leqq \sum_1^{p+1}{}_{\varkappa}\, \{B_\kappa\} = \omega - m.$$

But by the proposition just proved the order of the residual extension is $\omega - m$,

$$\therefore\ \{A'\} = \{B_1\},\quad \{B'_1\} = \{B_2\},\ \text{ etc. }\ \{B'_{p-1}\} = \{B_p\},\quad \{B'_{p+1}\} = \{B_{p+1}\},$$

and so $B'_{p+1} = B_{p+1}$.

Hence if C is any extension which has no part in common with the null region of ϕ, it is transformed by ϕ into an extension of the same order.

§10. The second branch of the law of nullity is easily derived from the two preceding sections. Denoting as before by A the null region of ϕ of nullity m, and by B the extension complementary to A with respect to the ground, let C denote the null-region of ψ of nullity n, and let $\mathfrak{B}$ denote that part of B which ϕ transforms into C. Obviously, the null-region of $\psi\phi$ is the aggregate of A, the region which ϕ annuls, and of $\mathfrak{B}$, that part of the ground complementary to A which ϕ transforms into C. By what was proved in §9 it follows that the order of $\phi\mathfrak{B}$ is the same as that of $\mathfrak{B}$. Whence to determine the order of $\mathfrak{B}$ it suffices to determine the order of $\phi\mathfrak{B}$, the extension common to C and the residual region of ϕ. The order of the residual region of ϕ is $\omega - m$ and the order of C is n; hence the extension $\phi\mathfrak{B}$ common to C and the residual region of ϕ

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* It is obvious from the definition of a linear unit operator that it cannot increase the order of an extension; so if $\phi A = B$, hence $\{B\} \leqq \{A\}$.

is at most of order n. If $(\omega - m) + n > \omega$, i. e. $m < n$, the extension C and the residual region of ϕ have common an extension of order at least $((\omega - m) + n) - \omega = n - m$, but if $(\omega - m) + n \leqq \omega$, i. e. $m \geqq n$, these two extensions do not necessarily have any part common. Hence the order of $\phi\mathfrak{B}$, and consequently of $\mathfrak{B}$, is at most n; and if $m < n$, the order of $\mathfrak{B}$ is not less than $n - m$, but if $m \geqq n$, the order of $\mathfrak{B}$ may be zero. The extension A is of order m and has no part in common with $\mathfrak{B}$. Hence the null-region of $\psi\phi$ cannot be of order greater than $m + n$; and if $m < n$, it cannot be less than $(n - m) + m = n$, but if $m \geqq n$, it cannot be less than m; and, consequently, the nullity of *the product of two matrices is not greater than the sum of their nullities nor less than the greater nullity of the two matrices.** This theorem is due to Prof. Sylvester, who terms it the law of nullity. Owing to the importance of the relation of the null-extension to the nullity of a matrix, I term the whole relation of the null-extension and nullity of one or more matrices the law of nullity, and this theorem the second branch of the law.

Suppose that C, the null-region of ψ, has no part in common with the vacuous region of ϕ; let E denote the extension common to C and A, the null region of ϕ; then the residue of C, namely $C - E$, is wholly contained in that part of the ground which is complementary to the aggregate of the null-region of ϕ and the vacuous region of ϕ, and which was denoted in §9 by B_{p+1}. Let $\mathfrak{B}_1$ denote that part of the ground complementary to A which is transformed by ϕ into E ($\mathfrak{B}_1$ is evidently contained in B_1); let the order of E be p and the order of $\mathfrak{B}_1$ be q. Since $\phi B_{p+1} = B_{p+1}$ (§9), the null-region of $\psi\phi$ is obviously the aggregate A, of $\mathfrak{B}_1$, and of $C - E$, which are all mutually exclusive; and, consequently the order of the null-region of $\psi\phi$ is $m + q + (n - p)$. But $p \leqq m$, $p \leqq n$; moreover, the order of the extension $\phi\mathfrak{B}_1$ is equal to that of $\mathfrak{B}_1$, but $\phi\mathfrak{B}_1$ is included in E, hence $q \leqq p$. If $p = 0$, the order of null-region of $\psi\phi$ is $m + n$. In this case the null-region of ψ is included wholly in that part of the ground complementary to the aggregate of the null-region of ϕ and the vacuous region of ψ. The aggregate of these two extensions I shall term, for reasons which

*The outline of this proof was communicated by me to the Johns Hopkins University Mathematical Society in a paper read before the Society, Nov. 1888, when I was informed by Dr. Franklin that this method of proving Sylvester's law had been employed by Mr. Bucheim in a note in the Phil. Mag. (XVIII, 459). I had previously had no knowledge of any anticipation of this method of dealing with the theory of matrices except what is contained in Clifford's "Fragment on Matrices." Subsequently I examined Mr. Bucheim's proof, and I find that though it is sufficient to give the limits between which the nullity of the product of two matrices lies, nevertheless it is defective in assuming what is not always true, viz. that the resultant region of a matrix is coincident with the extension which in §9 I have denoted by B_{p+1}. Mr. Bucheim also omits to prove that the order of $\mathfrak{B}$ is the same as that of $\phi\mathfrak{B}$.

will appear presently, the latent region of ϕ corresponding to its latent root zero. *Hence if the null-region of* ψ *has no part in common with the latent region of* ϕ *corresponding to the latent root zero, the nullity of* $\psi\phi$ *is the sum of the nullities of* ϕ *and* ψ. *A fortiori*, if the latent regions of ϕ and ψ corresponding to the latent root zero are mutually exclusive, the nullity of $\psi\phi$ (and also of $\phi\psi$) is the sum of the nullities of ϕ and ψ.

If the order of the vacuous region of ϕ is zero, so that the null-region of ϕ is coextensive with the latent region of ϕ corresponding to the latent root zero, the null-region of $\psi\phi$ is the aggregate of A and C; hence if these extensions do not intersect, the nullity of $\psi\phi$ is $m+n$. Conversely, *if the order of the vacuous region of* ϕ *is zero, and the nullity of* $\psi\phi$ *is* $m+n$, *then the null-regions of* ϕ *and* ψ *have no part in common.*

Nullity of the Factors of the Identical Equation.

§11. If the latent roots of ϕ are all distinct, the nullity of the product $\Phi_1 = (\phi - g_1)(\phi - g_2) \ldots . (\phi - g_m)$ of m factors of the identical equation is m.* This theorem is Prof. Sylvester's, and is termed by him *the corollary of the law of nullity.* His demonstration is as follows: Since each factor, being simply vacuous, has a nullity of order unity, the nullity of the product Φ_1 cannot exceed m. Similarly the nullity of the product Φ_2 of the remaining factors of the identical equation cannot exceed $\omega - m$. But the nullity of the product of Φ_1 and Φ_2 is ω, and consequently the sum of their nullities must be as great as ω. Hence the nullity of Φ_1 cannot fall short of m, and the nullity of Φ_2 cannot fall short of $\omega - m$.

When the latent roots of ϕ are not all distinct, the law is not so simple. Suppose the distinct latent roots of ϕ be i in number, namely, $g_1, g_2, \ldots . g_i$, occurring severally $m_1, m_2, \ldots . m_i$ times, and that the identical equation is

$$(\phi - g_1)^{m_1 - \kappa_1} (\phi - g_2)^{m_2 - \kappa_2} \ldots . (\phi - g_i)^{m_i - \kappa_i} = 0.$$

For the investigation of this case the following lemma is required, namely, that the vacuity of any positive integer power of $\phi - g$, for any latent root g, is equal to the vacuity of $\phi - g$, and hence to the number of times the latent root g occurs. This lemma is an immediate consequence of the theorem that the determinant of the product of two matrices is the product of their determinants;

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* From the law of latency, §15, it follows that the vacuity of Φ_1 is also m.

for then, if m is a positive integer, if μ is a primitive m^{th} root of unity, and if $\psi = \phi - g$,

$$|\psi^m - g^m| \equiv |\psi - g| \, . \, |\psi - \mu g| \, . \, |\psi - \mu^2 g| \, \ldots \, |\psi - \mu^{m-1} g| \, .$$

But if g^r is the lowest power of g that appears in the latent function $|\psi - g|$ of ψ, then $(\mu g)^r$ is the lowest power of μg that appears in $|\psi - \mu g|$, etc.; whence the lowest power of g^m that appears in $|\psi^m - g^m|$ is the r^{th} power of g^m. Consequently $\psi^m = (\phi - g_1)^m$ has the same vacuity as $\psi = \phi - g_1$.* Resuming the investigation of the nullity of any product of matrices $\phi - g$ for different latent root g, as a consequence of this lemma, the several factors $(\phi - g_1)^{m_1 - \kappa_1}$, $(\phi - g_2)^{m_2 - \kappa_2}$, etc., of the identical equation have the vacuities m_1, m_2, etc.; and hence their several nullities cannot exceed m_1, m_2, etc., respectively. But the nullity of their product, which cannot exceed the sum of their nullities, is ω. Hence the nullities of the several factors are m_1, m_2, etc., respectively; and hence corresponding to each latent root g of ϕ is an extension of order equal to the number of times that latent root is repeated which is annulled by that power of $\phi - g$ appearing in the identical equation.† These extensions I term the latent extensions or *latent regions* of the latent roots. In a similar way it may be shown that the nullity of the product of any two factors of the identical equation $(\phi - g_1)^{m_1 - \kappa_1}$ and $(\phi - g_2)^{m_2 - \kappa_2}$ is $m_1 + m_2$; but since the vacuity of either matrix is equal to its nullity, the order of the vacuous region of either matrix is zero; and hence by §10 their null-regions have no part in common. Similarly with respect to the null-regions of any two other factors of the identical equation. Hence the latent regions are mutually exclusive.‡ Since each positive integer power of $\phi - g_1$ is vacuous, there is, evidently, an extension of order at least unity annulled by $\phi - g_1$; an extension annulled by $(\phi - g_1)^2$, etc.; and each of these extensions is included in the one corresponding to the next higher power of $\phi - g_1$ in the series, since if an extension is annulled by any power of $\phi - g_1$ it is annulled by the next higher power. But as the nullity of no power of $\phi - g_1$ is greater than m_1, which is the order of the latent region corresponding to g_1, all these extensions are included in this latent region. Similarly with respect to the other latent roots. The latent region of

* It should be remembered that the latent roots of a matrix are the roots of its latent function.

† In this method of finding the nullity of the several factors $(\phi - g_1)^{m_1 - \kappa_1}$ of the identical equation, I have followed another one of Sylvester's methods of demonstrating the corollary of the law of nullity when the latent roots are all distinct, the only case Sylvester considers.

‡ That the latent extensions are mutually exclusive may be proved very simply by the method employed in the first part of §12.

ϕ corresponding to the latent roots g_1, g_2, etc., are, however, obviously the respective latent regions corresponding to the latent root zero of the matrices $\phi - g_1$, $\phi - g_2$, etc.; and consequently these extensions are also respectively the latent regions corresponding to the latent root zero of any positive integer powers of $\phi - g_1$, $\phi - g_2$, etc. Whence the nullity of the product whose factors are powers of $\phi - g_1$, $\phi - g_2$, etc., is by §10 the sum of the nullities of the several factors, as the latent regions of these factors corresponding to the latent root zero are mutually exclusive.

The next problem is to find the nullity of successive powers of ϕ less than any of its latent roots; and I shall show that the nullity of the $(\phi - g_1)^2$ is greater than the nullity of $\phi - g_1$ by an amount at least unity (unless the nullity of $\phi - g_1$ is m_1), and that the nullity of successive powers of $\phi - g_1$ goes on increasing by an amount not greater than the increment of nullity in the preceding power, until some power of $\phi - g_1$ is reached whose nullity is equal to its vacuity; that power of $\phi - g_1$ whose vacuity is equal to its nullity is evidently the factor in the identical equation corresponding to the latent root g_1. This theorem has already been proved in §9, where it was shown that the region annulled by $(\phi - g_1)^2$ consisted of A the null region of $\phi - g_1$, and of B_1 the extension transformed by $\phi - g_1$ into A, of order equal to or less than that of A, etc. The theorem may also be proved as follows: Denote $\phi - g_1$ by ψ, let the null region of ψ of order a be denoted by A; and, of the complementary extension, let B_1 of order b be that part transformed by ψ into A. Such an extension must exist, otherwise ψ^2 would annul only the extension A; hence A would also be the null region of ψ^3, since ψ^3 can annul only the null region of ψ^2 together with that extension projected into it by ψ, etc. Finally, no power of ψ would annul an extension other than A; but then the factor ψ would occur in the identical equation only to the first power; consequently the nullity of ψ would be as great as its vacuity. The order of B_1 cannot be greater than the order of A, for then the ψ of any b linearly independent quantities $\beta_1, \beta_2, \ldots \beta_b$, in the region B_1 would be expressible linearly in terms of $a < b$ linearly independent quantities in the region A; hence for some value of the t's other than all zero, the expression

$$t_1\psi\beta_1 + t_2\psi\beta_2 + \ldots + t_b\psi\beta_b = \psi(t_1\beta_1 + t_2\beta_2 + \ldots + t_b\beta_b)$$

would vanish, which is contrary to supposition, since no part of B_1 is in the null region of ψ. In the same way it may be shown that if the nullity of ψ^{m+1} is not greater than the nullity of ψ^m, no higher power of ψ has a nullity greater than the nullity of ψ^m, which must therefore be equal to its vacuity; and also that the

extension annulled by ψ^{n+2} additional to that annulled by ψ^{n+1} cannot be of order greater than the order of the extension annulled by ψ^{n+1} additional to the null region of ψ^n.

§12. No rational integral function of ϕ operating upon any quantity in the latent region corresponding to one latent root can transform it wholly or in part into the latent region of any other latent root. By Sylvester's formula, §16, it may be shown that any function of ϕ may be reduced to a rational integral function; whence the above proposition may be stated for any function of ϕ. It is obviously only necessary to prove that no function of ϕ can transform any quantity in one latent region wholly into the latent region of another latent root. Suppose $F\phi$ is a rational integral function of ϕ of order n; let ξ_1 be a quantity in the region of g_1. If $F\phi.\xi_1 = \xi_2$, where ξ_2 is a quantity in the latent region of g_2, then for some integer $m \geqq m_2 - \varkappa_2$,

$$((\phi - g_1)^m + (g_1 - g_2)(\phi - g_1)^{m-1} + \dots + (g_1 - g_2)^{m-1}(\phi - g_1) + (g_1 - g_2)^m) F\phi\xi_1 = (\phi - g_2)^m \xi_2 = 0.$$

Suppose $F\phi$ contains $\phi - g_1$ as a factor to the p^{th} power, then unless $F\phi\xi_1 = 0$, a case which need not be considered since ξ_2 is to be regarded as non-evanescent, there exists a linear relation

$$(\phi - g_1)^{m+n}\xi_1 + A(\phi - g_1)^{m+n-1}\xi_1 + \dots + L(\phi - g_1)^{p+1}\xi_1 + M(\phi - g_1)^p\xi_1 = 0,$$

where $M \neq 0$, between $(\phi - g_1)^p\xi_1$ and other quantities that are annulled by successive powers of $\phi - g_1$, which is impossible.

Since the latent regions are mutually exclusive and together make up the extension of the ground, a number of linearly independent quantities may be taken from each latent region and together may be employed to represent the ground.

By what has just been proved, each latent region in respect to ϕ constitutes a subordinate ground; for the effect of the matrix ϕ upon any one of these extensions is to convert a set of linearly independent quantities, equal in number to its order, into other quantities in the same extension. It would thus seem that ϕ might be regarded as an aggregate of subordinate matrices corresponding to and equal in number to the distinct latent roots, each of which would have upon the latent region corresponding to it the effect of ϕ, and would have a null effect upon any other extension. This suggestion may be verified as follows: Let the latent roots of ϕ be i in number, namely, $g_1, g_2 \dots g_i$, occurring severally $m, n, \dots p$ times; let the latent regions corresponding to the g's be $A, B, \dots L$

respectively; and, finally, let any m linearly independent quantities $\alpha_1, \alpha_2, \ldots \alpha_m$ be selected from A, any n linearly independent quantities $\beta_1, \beta_2, \ldots. \beta_n$ be selected from B, etc., and let any p linearly independent quantities $\lambda_1, \lambda_2, \ldots \lambda_p$ be selected from L; these quantities together embrace the extension of the ground. If

$$\begin{aligned}
&\phi\alpha_1 = a_{11}\alpha_1 + a_{21}\alpha_2 + \ldots. + a_{m1}\alpha_m,\\
&\phi\alpha_2 = a_{12}\alpha_1 + a_{22}\alpha_2 + \ldots. + a_{m2}\alpha_m, \text{ etc.,}\\
&\phi\beta_1 = b_{11}\beta_1 + b_{21}\beta_2 + \ldots. + b_{n1}\beta_n,\\
&\phi\beta_2 = b_{12}\beta_1 + b_{22}\beta_2 + \ldots. + b_{n2}\beta_n, \text{ etc.,}\\
&\text{etc., etc.,}\\
&\phi\lambda_1 = l_{11}\lambda_1 + l_{21}\lambda_2 + \ldots. + l_{p1}\lambda_p,\\
&\phi\lambda_2 = l_{12}\lambda_1 + l_{22}\lambda_2 + \ldots. + l_{p2}\lambda_p, \text{ etc.,}
\end{aligned}$$

then the form of ϕ becomes

$$\left|\begin{array}{c|c|c|c}
\begin{matrix} a_{11} & a_{12} & \ldots. & a_{1m}\\ a_{21} & a_{22} & \ldots. & a_{2m}\\ \ldots & \ldots & \ldots & \ldots\\ a_{m1} & a_{m2} & \ldots. & a_{mm}\end{matrix} & & & \\
\hline
 & \begin{matrix} b_{11} & b_{12} & \ldots. & b_{1n}\\ b_{21} & b_{22} & \ldots. & b_{2n}\\ \ldots & \ldots & \ldots & \ldots\\ b_{n1} & b_{n2} & \ldots. & b_{nn}\end{matrix} & & \\
\hline
 & & \ddots & \\
\hline
 & & & \begin{matrix} l_{11} & l_{12} & \ldots. & l_{1p}\\ l_{21} & l_{22} & \ldots. & l_{2p}\\ \ldots & \ldots & \ldots & \ldots\\ l_{p1} & l_{p2} & \ldots. & l_{pp}\end{matrix}
\end{array}\right|$$

In this form all the constituents except those in the squares along the diagonal are zero. It is obvious from this form that the array of the a's forms a matrix by itself, and that its effect upon any region other than that of the latent root g_1 is to annul it, etc. Whence if ϕ_1 denote the matrix formed from the a's, and ϕ_2 that formed from the b's, etc., then ϕ_1, ϕ_2, etc., are nil-factorial together, and

$$\phi = \phi_1 + \phi_2 + \dots + \phi_i,$$
$$\therefore \phi^2 = \phi_1^2 + \phi_2^2 + \dots + \phi_i^2;$$

and, in general, $$F\phi = F\phi_1 + F\phi_2 + \dots + F\phi_i.$$

The matrices ϕ_1, ϕ_2, etc., regarded as pertaining to the ground of ϕ, are, of course, vacuous; thus, from this point of view, ϕ_1 has the representation

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} & 0 & \dots & 0 \\ a_{21} & a_{22} & \dots & a_{2m} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mm} & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \end{pmatrix}$$

and has a nullity at least $\omega - m$; if ϕ is non-vacuous, the nullity of ϕ_1 is just $\omega - m$, etc. But regarded as subordinate matrices of different systems, pertaining to the subordinate grounds A, B, $\dots$ L, then, unless ϕ is vacuous, ϕ_1, ϕ_2, $\dots$ ϕ_i must be considered as non-vacuous matrices; and ϕ_1 will have the representation

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mm} \end{pmatrix}_A,$$

this matrix being supposed to operate only upon that part of the ground comprised in the subordinate ground A, namely, the extension of the set $(\alpha_1, \alpha_2, \dots \alpha_m)$, and must be regarded as not susceptible of operating upon any other ground. The subscript A is employed to denote that the ultimate operands of this matrix are only expressions linear in the α's, etc. From this point of view it is proper to consider ϕ_1, ϕ_2, etc., as having reciprocals, providing ϕ is not vacuous; and then

$$\phi^{-1} = \phi_1^{-1} + \phi_2^{-1} + \dots + \phi_i^{-1}.$$

If ϕ is vacuous it must have zero as a latent root; but only one of the subordinate matrices corresponds to this root; consequently one and only one of the subordinate matrices will be vacuous, and evidently the vacuous subordinate matrix will have exactly the same vacuity and nullity as ϕ. The vacuous subordinate matrix is evidently a nilpotent quantity, since if ϕ is raised to a sufficiently higher power it annuls the region corresponding to the latent root zero. Thus if $g_1 = 0$, then A is annulled by $\phi^{m_1-\kappa_1}$; hence

$$\phi^{m_1-\kappa_1} = 0 + \phi_2^{m_1-\kappa_1} + \ldots + \phi_i^{m_1-\kappa_1}.$$

The conception of a matrix as a sum of a set of subordinate matrices is more readily grasped by the consideration of Peirce's linear-form representation of a matrix. Thus

$$\phi = \Sigma_r\Sigma_s a_{rs}(\alpha_r:\alpha_s) + \Sigma_r\Sigma_s b_{rs}(\beta_r:\beta_s) + \ldots + \Sigma_r\Sigma_s l_{rs}(\lambda_r:\lambda_s).$$

But evidently the set of vids $(\alpha_r:\alpha_s)$ form a quadrate system by themselves, similarly with respect to the set of vids $(\beta_r:\beta_s)$, etc., and any linear expressions in the vids $(\alpha_r:\alpha_s)$ will have as ultimate operands only expressions linear in the α's; while any expression linear in two different sets of vids, if they are to be regarded as susceptible of operating upon each other, are mutually nil-facient and nil-faciend.

Considering the vids $(\alpha_r:\alpha_s)$ as a quadrate system by themselves, scalar unity will be expressed by $\Sigma_1^m r(\alpha_r:\alpha_r)$, and may be denoted by 1_1. The quadrate system formed from the vids $(\beta_r:\beta_s)$ will have as its scalar unity $\Sigma_1^n r(\beta_r:\beta_r)$, which may be denoted by 1_2, etc. We evidently have

$$1_1 1_2 = 1_2 1_1 = 1_1 1_3 = 1_3 1_1 = \text{etc.} = 0;$$

and, denoting by 1 the unity of the complete system,

$$1 = 1_1 + 1_2 + \ldots + 1_i.$$

Thus $$\phi - g = (\phi_1 - g1_1) + (\phi_2 - g1_2) + \ldots + (\phi_i - g1_i).$$

Since any matrix can thus be resolved into an aggregate of as many other matrices (mutual nil-factorial), as it has latent roots, each subordinate matrix corresponding to a latent root and being of order equal to the number of times that latent root occurs, hence in general it suffices to prove a theorem

relating to a single matrix for one all of whose latent roots are equal, when, if true, it may by means of this proposition be inferred for any case.*

§13. Having obtained the law governing the nullity of the factors of the identical equation, it is now possible to solve the problem touched upon in §3 with regard to a matrix subject to a condition involving only itself. As was stated above, it follows from Sylvester's formula that any condition equation to which a matrix is subject may be reduced to a rational integral equation. Let

$$F\phi \equiv (\phi - g_1)^{\alpha}(\phi - g_2)^{\beta} \ldots . (\phi - g_m)^{\sigma} \ldots . (\phi - g_n)^{\tau} = 0$$

be the rational integral equation expressing the condition to which ϕ is subject; then ϕ will satisfy this condition if any $m \leqq \omega$ of the g's are selected as its latent roots, as $g_1, g_2, \ldots . g_m$, each occurring $\alpha', \beta', \ldots . \sigma'$ times respectively; and if $(\phi - g_1)^{\alpha' - \kappa_1}$ has the nullity α', $(\phi - g_2)^{\beta' - \kappa_2}$ the nullity β', etc., and $(\phi - g_m)^{\sigma' - \kappa_m}$ has the nullity σ', provided the sum of the accented Greek letters is ω, and $\alpha' - \kappa_1 \leqq \alpha$, $\beta' - \kappa_2 \leqq \beta$, etc. For then

$$(\phi - g_1)^{\alpha' - \kappa_1}(\phi - g_2)^{\beta' - \kappa_2} \ldots . (\phi - g_m)^{\sigma' - \kappa_m} = 0,$$

$$\therefore\ (\phi - g_1)^{\alpha}(\phi - g_2)^{\beta} \ldots . (\phi - g_m)^{\sigma} \ldots . (\phi - g_n)^{\tau} = 0.$$

Conversely, if ϕ satisfies the last equation it must have a certain number of the g's as latent roots, each occurring a sufficient number of times to make the total number of latent roots equal to ω; and if g is any one of these latent roots, some power of $\phi - g$ equal to or lower than that which appears in the above equation must have a nullity equal to the number of times g occurs.

Law of Congruity and Law of Latency.

§14. If ρ_1 is an axis of ϕ, it is an axis of any integer power ϕ^m of ϕ; for if $\phi\rho_1 = g_1\rho_1$, hence $\phi^m\rho_1 = g_1^m\rho_1$. Similarly with respect to the other axes of ϕ. Hence the axes of ϕ are all axes of ϕ^m; but since the axes of ϕ may be linearly related, they do not necessarily constitute all the axes of ϕ^m. In like manner the axes of any root $\phi^{\frac{1}{n}}$ (where n is an integer) are comprised among the axes of ϕ. We have for each latent root g_r of ϕ, v being a primitive n^{th} root of unity, the equation

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*Thus if the identical equation has been proved for a matrix of any order whose latent roots are all equal, it may be shown by the principle of this section to hold in any case. For if

$$\phi = \phi_1 + \phi_2 + \ldots . + \phi_i,$$

and

$$0 = (\phi_1 - g_1 1_1)^{m_1} = (\phi_2 - g_2 1_2)^{m_2} = \text{etc.},$$

$$\therefore\ (\phi - g_1)^{m_1}(\phi - g_2)^{m_2} \ldots . (\phi - g_i)^{m_i} = 0.$$

$$(\phi^{\frac{1}{n}} - g_r^{\frac{1}{n}})(\phi^{\frac{1}{n}} - vg_r^{\frac{1}{n}})(\phi^{\frac{1}{n}} - v^2g_r^{\frac{1}{n}}) \ldots . (\phi^{\frac{1}{n}} - v^{n-1}g_r)\,\rho_r = (\phi - g_r)\,\rho_r = 0\,.$$

Evidently one at least of the matrices which in the first member operate upon ρ_r must be vacuous, and hence one of the n^{th} roots of each distinct latent roots of ϕ is a latent root of $\phi^{\frac{1}{n}}$. If then the latent roots g_1, g_2, etc., of ϕ are all distinct, so also are the latent roots of $\phi^{\frac{1}{n}}$; and then, by §2, the axes of ϕ, ρ_1, ρ_2, etc., are ω linearly independent and determinate quantities, as are also the axes of $\phi^{\frac{1}{n}}$; and since the latter are comprised among the axes of ϕ, it is evident that every axis of ϕ is an axis of $\phi^{\frac{1}{n}}$. Hence every axis of ϕ is an axis of $\phi^{\frac{m}{n}}$; and for the ω axes we have the equations $\phi^{\frac{m}{n}}\rho_1 = g_1^{\frac{m}{n}}\rho_1$, $\phi^{\frac{m}{n}}\rho_2 = g_2^{\frac{m}{n}}\rho_2$, etc. It may now be established by the method of limits, in conformity with the definition of I, §8, that for any scalar m we have for the ω axes the equations $\phi^m\rho_1 = g_1^m\rho_1$, $\phi^m\rho_2 = g_2^m\rho_2$, etc. Consequently $\Sigma k\phi^m.\rho_1 = \Sigma kg_1^m.\rho_1$, $\Sigma k\phi^m\rho_2 = \Sigma kg_2^m.\rho_2$, etc., provided the coefficients and exponents are scalars; and so if $F\phi$ is any function of ϕ formed by the addition of scalar multiples of scalar powers of ϕ, then for the ω axes $F\phi\rho_1 = Fg_1.\rho_1$, $F\phi\rho_2 = Fg_2.\rho_2$, etc. Thus what has been proved is, *when the latent roots of ϕ are all distinct, every axis of ϕ is an axis of $F\phi$*; and since the above equations, or their equivalent, $(F\phi - Fg_1)\,\rho_1 = 0$, $(F\phi - Fg_2)\,\rho_2 = 0$, etc., hold only for the latent roots of $F\phi$, hence *when the latent roots of ϕ are all distinct, the latent roots of $F\phi$ are the same function of the latent roots of ϕ.*

The first of these theorems I term the *law of congruity of the axes.* The second is very important; it is due to Prof. Sylvester, who terms it the *law of latency.* In the next section I shall extend the proof of the law of latency to the general case. The term law of congruity was employed by Prof. Sylvester interchangeably with the term law of latency.

The converse of the law of congruity is also true, subject to a slight modification when two or more of the latent roots of $F\phi$ become equal. In this case, if the axes of $F\phi$ are first selected, they may not all prove to be axes of ϕ; but when this occurs, other quantities may be chosen which, together with the axes common to both ϕ and $F\phi$, shall constitute the ω axes of $F\phi$ and also be axes of ϕ. Thus, to take an example, let ρ_1, ρ_2, ρ_3 be the three axes of the ternary

matrix ϕ whose corresponding latent roots are $1, -1, 2$ respectively; then the axis of ϕ^2 corresponding to the latent root unity is indefinite; for

$$\phi^2(x\rho_1 + y\rho_2) = x\rho_1 + y\rho_2 .$$

In general, this quantity will not be an axis of ϕ. However, ρ_1, ρ_2 and ρ_3 are axes of both ϕ and ϕ^2.

If the latent roots of ϕ are all distinct, the ground may be represented in terms of the axes of ϕ; in which case, if $F\phi$ is any function of ϕ involving only the matrix ϕ and unity, we have obviously

$$F\phi = \left(\begin{array}{cccc} Fg_1 & 0 & \dots & 0 \\ 0 & Fg_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & & Fg_\omega \end{array}\right)$$

From this form of $F\phi$ immediate proofs for the general case of the law of latency and of the law of congruity may be derived.

§15. If the latent roots of ϕ are all distinct, it is not necessarily true that the latent roots of $F\phi$ are all distinct. When this is not the case, the order of the identical equation in $F\phi$ is lowered by unity for each latent root of $F\phi$ which becomes equal to another. Thus suppose $Fg_1 = Fg_2 = \dots = Fg_n$, then $\rho_1, \rho_2, \dots \rho_n$ are all annulled by $F\phi - Fg_1$; consequently the application of this operator to the most general expression ρ in the ground (which can now, by §2, be expressed in terms of the ω axes of ϕ) annuls n components and leaves $\omega - n$ components which will be annulled by the remaining factors of the identical equation in $F\phi$. I. e. the degree of the identical equation is lowered by $n-1$.

The law of latency may be proved in the general case as follows: Let the latent roots of ϕ be all distinct, and let $g_2 - g_1 = h_2$, etc.; as h diminishes, $Fg_2 = Fg_1 + h_2F'g_1 +$ etc. approaches Fg_1. In the limit ϕ has two equal roots g_1, and $F\phi$ has two latent roots equal to Fg_1. Obviously, when ϕ has two or more equal latent roots, the latent roots of $F\phi$ corresponding to these are not necessarily equal, since $F\phi$ may be a many-valued function. Thus, if $(\phi - g_1)^2 = 0$, the latent roots of $\phi^{\frac{1}{2}}$ are $\pm\sqrt{g_1}$, and the identical equation in $\phi^{\frac{1}{2}}$ is

$$(\phi^{\frac{1}{2}} - \sqrt{g_1})(\phi^{\frac{1}{2}} + \sqrt{g_1}) = 0 .$$

Sylvester's Formula.

§16. Sylvester has given without demonstration the following theorem in the Johns Hopkins Univ. Circ. (No. 28, 1884):

$$F\phi = \Sigma F g_1 . \frac{(\phi - g_2)(\phi - g_3) \ldots . (\phi - g_\omega)}{(g_1 - g_2)(g_1 - g_3) \ldots . (g_1 - g_\omega)} .$$

It may be proved as follows: Suppose the latent roots of ϕ are all distinct, and denote for convenience the left-hand member of the above equation by (Σ); then for the ω axes of ϕ we have $(\Sigma)\rho_1 = Fg_1.\rho_1$, $(\Sigma)\rho_2 = Fg_2.\rho_2$, etc. But by the law of congruity and the law of latency, $F(\phi)\rho_1 = Fg_1.\rho_1$, $F(\phi)\rho_2 = Fg_2.\rho_2$, etc., for the ω axes; consequently $(\Sigma)\rho_1 = F(\phi)\rho_1$, $(\Sigma)\rho_2 = F(\phi)\rho_2$, etc.; and hence for any quantity linear in the axes $\rho = z_1\rho_1 + z_2\rho_2 +$ etc., $F(\phi)\rho = (\Sigma)\rho$. Since the latent roots of ϕ are all supposed distinct, ρ may be any quantity whatever in the ground of which ϕ is a linear unit function. Hence

$$F\phi = (\Sigma) .$$

As this mode of proof is a verification, if $F\phi$ is a many-valued function, it is necessary to show that Sylvester's formula gives all possible solutions of $\psi = F(\phi)$. Let the latent roots of ϕ be all distinct; take as latent roots of a matrix ψ any set of the values of Fg_1, Fg_2, etc., and as axes corresponding respectively to them the ω linearly independent axes of ϕ, ρ_1, ρ_2, etc.; then ψ satisfies the equation $\psi.\rho = F\phi.\rho$ for each of these axes, and consequently for any quantity in the ground. Evidently m^ω matrices ψ may thus be formed if m is the number of values of the function Fg, and only the matrices so formed satisfy for all values of ρ the condition $\psi\rho = F\phi.\rho$, and thus are solutions of $\psi = F\phi$. These m^ω matrices are, however, all contained in Sylvester's formula.

By the theory of limits the theorem may be extended to the case where the latent roots of ϕ are not all distinct.

In the general case in which the latent roots of ϕ are all distinct, a simple proof of Sylvester's formula may also be derived from the form of $F\phi$ when the axes of ϕ represent the ground. In §5 it was shown that when the axes of ϕ are linearly independent, whatever the set of quantities chosen to represent the ground, there is always a definite matrix ϖ such that

$$\phi = (\varpi) \begin{vmatrix} g_1 & 0 & \ldots . & 0 \\ 0 & g_2 & \ldots . & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots . & g_\omega \end{vmatrix} (\varpi^{-1}),$$

whence the form of ϕ being given in terms of any set of quantities, we can very simply find the form of $F\phi$ in terms of the same set of quantities, for

$$F\phi = (\varpi)\begin{vmatrix} Fg_1 & 0 & \dots & 0 \\ 0 & Fg_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & Fg_\omega \end{vmatrix}(\varpi)^{-1}.$$

If ϕ is a scalar, any quantity in the ground is an axis, hence the above considerations showing that Sylvester's formula gives all the solution of $\psi = F\phi$ do not apply. And on inspection it is evident that the formula does not give the non-scalar roots of a scalar. These must therefore be formed by an independent investigation. There are evidently two cases to be considered, the roots of the matrix zero and the roots of the matrix unity; from the latter the roots of any non-zero matrix may be found.

Roots of the Matrix Zero.

§17. It is very evident by §13 that the latent roots of a nilpotent quantity are all zero, and, conversely, that a matrix all of whose latent roots are zero is nilpotent. Whence it follows that the number of roots of zero with any index is infinite, meaning by index of a root of a matrix the least power of the root which will reproduce the matrix. Of the square roots of zero in a matrix of order ω there are $\omega^2 - 1$ linearly independent. Thus, if $\omega = 2$, the four vids of a dual matrix may be expressed linearly in terms of

$$(A:A) + (B:B),\ (A:A) - (B:B) + (A:B) - (B:A),\ (A:B) \text{ and } (B:A),$$

of which the first is unity and the other three square roots of zero. The proposition may be shown similarly for any value of ω.

No root of zero can have an index greater than ω. For, as Benjamin Peirce has shown, there is no linear relation possible between the powers of a nilpotent quantity that does not vanish; but if ϕ is a root of zero, unless its ω^{th} or some lower power vanishes, by means of the identical equation, the ω^{th} power of ϕ can be expressed linearly in terms of the lower powers, the coefficients not all being zero.

If ϕ is an m^{th} root of zero, then by §11 the nullity of successive powers of ϕ increases until the m^{th} power is reached; and the nullity of ϕ^2 is at most twice

the nullity of ϕ, the nullity of ϕ^3 exceeds the nullity of ϕ^2 by an amount not greater than the increment of the nullity of ϕ^2 over the nullity of ϕ, etc.* Following the reasoning of §9, let the extension of the p linearly independent quantities $\alpha_1, \alpha_2, \ldots \alpha_p$ be the null region of ϕ; let the extension of the q linearly independent quantities $\beta_1, \beta_2, \ldots \beta_q$ be that transformed by ϕ into the null region of ϕ; let the extension of the r linearly independent quantities $\gamma_1, \gamma_2, \ldots \gamma_r$ be that transformed by ϕ into the extension of the β's, etc.; finally let the s linearly independent quantities $\varkappa_1, \varkappa_2, \ldots \varkappa_s$ be that transformed by ϕ into the extension annulled by ϕ^{m-2} additional to that annulled by lower powers of ϕ; and let the extension of the t linearly independent quantities $\lambda_1, \lambda_2, \ldots \lambda_t$ constitute the remaining extension of the ground. Then these sets of quantities, the α's, β's, etc., together embrace the extension of the ground, and

$$\phi\alpha_1 = \phi\alpha_2 = \ldots = \phi\alpha_r = 0,$$

$$\begin{aligned}
\phi\beta_1 &= a_{11}\alpha_1 + a_{21}\alpha_2 + \ldots + a_{p1}\alpha_p, \\
\phi\beta_2 &= a_{12}\alpha_1 + a_{22}\alpha_2 + \ldots + a_{p2}\alpha_p, \\
&\ldots\ldots\ldots\ldots\ldots\ldots \\
\phi\beta_q &= a_{1q}\alpha_1 + a_{2q}\alpha_2 + \ldots + a_{pq}\alpha_p,
\end{aligned}$$

$$\begin{aligned}
\phi\gamma_1 &= b_{11}\beta_1 + b_{21}\beta_2 + \ldots + b_{q1}\beta_q, \\
\phi\gamma_2 &= b_{12}\beta_1 + b_{22}\beta_2 + \ldots + b_{q2}\beta_q, \\
&\ldots\ldots\ldots\ldots\ldots\ldots \\
\phi\gamma_r &= b_{1r}\beta_1 + b_{2r}\beta_r + \ldots + b_{qr}\beta_q,
\end{aligned}$$

etc.; finally,

$$\begin{aligned}
\phi\lambda_1 &= k_{11}\varkappa_1 + k_{21}\varkappa_2 + \ldots + k_{s1}\varkappa_s, \\
\phi\lambda_2 &= k_{12}\varkappa_1 + k_{22}\varkappa_2 + \ldots + k_{s2}\varkappa_s, \\
&\ldots\ldots\ldots\ldots\ldots\ldots \\
\phi\lambda_t &= k_{1t}\varkappa_1 + k_{2t}\varkappa_2 + \ldots + k_{st}\varkappa_s.
\end{aligned}$$

* Hence certain roots of nilpotent quantities have no representation. Thus let ϕ be a matrix of order six whose nullity is two, and whose fourth power and no lower power vanishes; then ϕ has no square root. For if $\psi^2 = \phi$, then, since ψ has at least simple nullity, the nullity of $\psi^6 = \phi^3$ is at least six: hence $\phi^3 = 0$.

Whence the form now representing ϕ is

$$\left|\begin{array}{ccccc}
 & \begin{matrix} a_{11} & a_{12} & \cdots & a_{1q} \\ a_{21} & a_{22} & \cdots & a_{2q} \\ \cdots & \cdots & \cdots & \cdots \\ a_{p1} & a_{p2} & \cdots & a_{pq} \end{matrix} & & & \\
 & & \begin{matrix} b_{11} & b_{12} & \cdots & b_{1r} \\ b_{21} & b_{22} & \cdots & b_{2r} \\ \cdots & \cdots & \cdots & \cdots \\ b_{q1} & b_{q2} & \cdots & b_{qr} \end{matrix} & & \\
 & & & \ddots & \\
 & & & & \begin{matrix} k_{11} & k_{12} & \cdots & k_{1t} \\ k_{21} & k_{22} & \cdots & k_{2t} \\ \cdots & \cdots & \cdots & \cdots \\ k_{s1} & k_{s2} & \cdots & k_{st} \end{matrix} \\
 & & & &
\end{array}\right|$$

In this form all the constituents except those in the rectangles bordered on the left and below by heavy lines are zero. The successive heavy lines at every other alternate corner intersect with the diagonal, and no one of them is longer than the next preceding one. The square at the left-hand upper corner and the square at the lower right-hand corner consist respectively of p^2 and t^2 zeros.

If ϕ is a root of zero, so also is its converse.

Roots of the Matrix Unity.

§18. If ϕ is an m^{th} root of unity it is subject to the condition

$$\phi^m - 1 \equiv (\phi - \lambda_1)(\phi - \lambda_2) \dots (\phi - \lambda_m) = 0,$$

the λ's being the scalar m^{th} roots of unity. By §13, if any $n \leqq \omega$ of the m λ's, as $\lambda_1, \lambda_2, \dots \lambda_n$ repeated respectively $k_1, k_2, \dots k_n$ times (the sum of the k's being ω), are chosen as the latent roots of ϕ; and if $\phi - \lambda_1, \phi - \lambda_2, \dots \phi - \lambda_n$ have the respective nullities $k_1, k_2, \dots k_n$, then ϕ satisfies the condition; and conversely. It is obvious that the number of the non-scalar roots of unity with any index is infinite.* If no lower power of ϕ than the m^{th} is a scalar, then ϕ will be termed a *primitive* m^{th} root of unity. The condition necessary and sufficient that ϕ, in addition to being an m^{th} root of unity, shall be a primitive m^{th} root, is that one at least of its latent roots shall be primitive.

When the index $m \leqq \omega$, and the entire set of the scalar m^{th} roots of unity are latent roots of ϕ, then any m successive powers of ϕ are linearly independent. For then the identical equation is $\phi^m - 1 = 0$; but if there were any linear relation between m successive powers of ϕ, the identical equation would be of order $m - 1$, which, by §2, is impossible if ϕ has m distinct latent roots.

In the calculus of matrices, as in ordinary algebra, the m^{th} roots of any matrix may in general be obtained by taking any one of its m^{th} roots and multiplying it successively by all the m^{ω} m^{th} roots of unity of a certain set, namely, those that have as axes the axes of the matrix.

It is obvious that if ϕ is an m^{th} root of unity, so also is its converse, and if ϕ is primitive, so also is $\breve{\phi}$.

§19. *Every matrix whose order is a prime number has ω^2 linearly independent, primitive, ω^{th} roots of unity.* For, ω being prime, let ϕ denote the matrix

---

*If ϕ is not a scalar and has all its latent roots distinct, as was stated in §16, $\phi^{\frac{1}{m}}$ (where m is an integer) has in general m^{ω} values, obtained by combining each axis of ϕ with one of the m^{th} roots of the latent root of ϕ corresponding to that axis. The non-scalar roots of a scalar may similarly be obtained, but since any quantity is the axis of a scalar, these roots are infinite in number.

$$
\begin{array}{c|ccccccccccc|}
 & (1) & (2) & & (\omega-\varkappa) & (\omega-\varkappa+1) & (\omega-\varkappa+2) & (\omega-\varkappa+3) & & (\omega-1) & (\omega) \\
\hline
(1) & 0 & 0 & \ldots & 0 & 0 & a_1 & 0 & \ldots & 0 & 0 \\
(2) & 0 & 0 & \ldots & 0 & 0 & 0 & a_2 & \ldots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
(\varkappa-2) & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & a_{\varkappa-2} & 0 \\
(\varkappa-1) & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 & a_{\varkappa-1} \\
\varkappa & a_\varkappa & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\
(\varkappa+1) & 0 & a_{\varkappa+1} & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
(\omega-1) & 0 & 0 & \ldots & a_{\omega-1} & 0 & 0 & 0 & \ldots & 0 & 0 \\
(\omega) & 0 & 0 & \ldots & 0 & a_\omega & 0 & 0 & \ldots & 0 & 0 \\
\hline
\end{array}
$$

in which all the constituents are zero except one in each column, and the non-zero constituents, forming a broken diagonal, are denoted each by a with subscript indicating the row in which it appears; the constituent in the $\varkappa^{\text{th}}$ row appears in the first column, that in the $(\varkappa+1)^{\text{th}}$ row in the second column, and, in general, if $[x]$ denotes the smallest positive residue (modulus ω) of x, then the constituent in the $[\varkappa+r-1]^{\text{th}}$ rows appears in the r^{th} column. The constituent in the first column I shall term the leading constituent. It is evident, then, if $(\alpha_1, \alpha_2, \ldots \alpha_\omega)$ are the elements of the set upon which ϕ operates,

$$\phi\alpha_1 = a_\varkappa\alpha_\varkappa, \quad \phi\alpha_2 = a_{\varkappa+1}\alpha_{\varkappa+1}, \quad \text{etc.},$$

and that the general expression for the ϕ of any one α_r of the α's is

$$\phi\alpha_r = a_{[\varkappa+r-1]}\alpha_{[\varkappa+r-1]}.$$

I. e. ϕ applied to the α's advances each by $\varkappa-1$ places and multiplies it by a certain one of the a's. The application of ϕ^2 to each of the α's advances it by $2(\varkappa-1)$ places, and multiplies it by the product of two of the a's, etc.; finally, ϕ^ω advances each α by a multiple of ω places, i. e., transforms each α into itself and multiplies it by the product of all the a's. If, now, λ be an imaginary ω^{th} root of unity, and the successive a's, $a_1, a_2, \ldots a_\omega$ are severally put equal to $1, \lambda, \lambda^2, \ldots \lambda^{\omega-1}$, denoting by ϕ_1 what ϕ then becomes, we have

$$\phi_1^\omega(\alpha_1, \alpha_2, \ldots \alpha_\omega) = \lambda^{\frac{\omega(\omega+1)}{2}}(\alpha_1, \alpha_2, \ldots \alpha_\omega).$$

Whence unless $\omega = 2$, ϕ_1 is an ω^{th} root of unity; and since ω is prime, it is readily seen that no lower power of ϕ_1 is a scalar. Similarly, denoting by ϕ_2 what ϕ becomes

when the a's in the same order are put equal to $1, \lambda^2, \lambda^4, \ldots \lambda^{2(\omega-1)}$, then ϕ_2 is also a primitive ω^{th} root of unity; finally, denoting by $\phi_{\omega-1}$ what ϕ becomes when for the a's in the same order are substituted the $(\omega-1)^{\text{th}}$ powers of $1, \lambda, \lambda^2, \ldots \lambda^{\omega-1}$, then $\phi_{\omega-1}^{\omega}=1$ and also no lower power of $\phi_{\omega-1}$ is a scalar. In this way $\omega-1$ matrices are obtained which are primitive ω^{th} roots of unity; besides these, another primitive ω^{th} root may be obtained by putting the a's all equal to unity, and this matrix may be denoted by ϕ_0. These ω matrices are linearly independent, for otherwise, for some set of values of the x's, other than all equal to zero, we should have

$$x_0\phi_0+x_1\phi_1+x_2\phi_2+\ldots+x_{\omega-1}\phi_{\omega-1}=0,$$

$$\begin{aligned}
\therefore\; x_0+x_1+x_2+\ldots+x_{\omega-1}&=0,\\
x_0+\lambda x_1+\lambda^2x_2+\ldots+\lambda^{\omega-1}x_{\omega-1}&=0,\\
x_0+\lambda^2x_1+\lambda^4x_2+\ldots+\lambda^{2(\omega-1)}x_{\omega-1}&=0,\\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots&\\
x_0+\lambda^{\omega-1}x_1+\lambda^{2(\omega-1)}x_2+\ldots+\lambda^{(\omega-1)^2}x_{\omega-1}&=0,
\end{aligned}$$

which is impossible. Thus any position of the leading constituent other than in the first row (i. e. for any value of $\varkappa$ other than $\varkappa=1$), a proper choice of the a's gives ω linearly independent ω^{th} roots of unity; and thus in this way $\omega(\omega-1)$ linearly independent ω^{th} roots of unity may be obtained; for it is evident that the matrices obtained for different values of $\varkappa$ are linearly independent, being expressions in different vids. In addition to these, let $\omega-1$ matrices be formed by taking as constituents of the principal diagonal in the first matrix the set of scalar $1, \lambda, \lambda^2, \ldots \lambda^{\omega-1}$; in the second, the squares of this set, etc., and finally, the $(\omega-1)^{\text{th}}$ powers of this set for the $(\omega-1)^{\text{th}}$ matrix, all the other constituents in each matrix being zero. These $\omega-1$ matrices are primitive ω^{th} roots of unity, as may easily be seen by considering their effect upon the set $(\alpha_1, \alpha_2, \ldots \alpha_\omega)$. That they are linearly independent of each other and unity may be shown by the above reasoning; and since they are formed from the vids along the diagonal, which do not occur in the other set of $\omega(\omega-1)$ roots of unity, they can have no linear relation with these. Thus, in addition to unity (or rather to a primitive scalar ω^{th} root of unity), ω^2-1 other primitive ω^{th} root of unity may be formed, all of which are linearly independent.

If $\omega=2$, the four matrices formed in this way are

$$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\quad \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\quad \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\quad \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}.$$

Of these the first is unity, the second and third are square roots of unity, and the fourth is a square root of -1. By multiplying the fourth root by $\sqrt{-1}$, it is of course converted into a square root of unity. Thus a dual matrix may also be expressed linearly in terms of four square roots of unity.

Algebras Analogous to Quaternions.

§20. It was stated in I, §10, that if i, j, k are any three mutually normal unit vectors, any quaternion may be expressed linearly in terms of the four new units

$$\frac{1+i\sqrt{-1}}{2}, \quad \frac{j+k\sqrt{-1}}{2}, \quad \frac{-j+k\sqrt{-1}}{2}, \quad \frac{1-i\sqrt{-1}}{2},$$

which, having the same multiplication table as the four vids of a dual matrix,

$$(A:A), \quad (A:B), \quad (B:A), \quad (B:B),$$

may, consequently, be regarded as respectively identical with them. This identification gives the following values for the ordinary quaternion units,

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad i = \begin{pmatrix} -\sqrt{-1} & 0 \\ 0 & \sqrt{-1} \end{pmatrix}, \quad j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad k = \begin{pmatrix} 0 & -\sqrt{-1} \\ -\sqrt{-1} & 0 \end{pmatrix}.$$

The discovery of this form of quaternions, which I have termed the canonical form of quaternions (I, §8), as has been stated, is due to Benjamin Peirce; it received its full significance only after the discovery by his son, Charles Peirce, of the unlimited system of quadrates formed from the system of vids $(A:A)$, $(A:B)$, etc., when it appeared that quaternions was only the first of this system of quadrate algebras, and the identification of quaternions with the theory of dual matrices was virtually accomplished. Evidently of all linear associative algebras the quadrate algebras form a class which are closely related, and consequently are closely analogous to quaternions. In the preceding section it was shown that all matrices or quadrates whose order is a prime number may be regarded as linear in unity and $\omega^2 - 1$ linearly independent, primitive ω^{th} roots of unity, just as quaternions is an algebra linear in unity and three square roots of -1, or of unity; whence the analogy between quaternions and these other quadrates extends beyond the quadrate form possessed by each. Moreover, these $\omega^2 - 1$ roots of unity are formed from the ω^2 vids of the quadrate of order ω, except for a scalar factor, precisely as the i, j, k of quaternions are formed

from the vids of the dual matrix. Thus the quadrate algebra of prime order next in order to quaternions (nonions) is that formed from the vids of a triple matrix: comparing its eight cube roots of unity with the i, j, k of quaternions, we have

$$i = [(A:A) - (B:B)]\sqrt{-1}, \quad j = (A:B) - (B:A), \quad k = [(A:B) + (B:A)]\sqrt{-1},$$

while the nine nonion cube roots are

$$\begin{array}{ll} (A:A) + \lambda(B:B) + \lambda^2(C:C), & (A:A) + \lambda^2(B:B) + \lambda(C:C), \\ (A:B) + (B:C) + (C:A)\quad, & (A:B) + \lambda(B:C) + \lambda^2(C:A), \\ (A:B) + \lambda^2(B:C) + \lambda(C:A)\ , & (A:C) + (B:A) + (C:B)\quad, \\ (A:C) + \lambda(B:A) + \lambda^2(C:B)\ , & (A:C) + \lambda^2(B:A) + \lambda(C:B)\ ;^* \end{array}$$

and the representation of unity in the first system is $(A:A) + (B:B)$, and in the second $(A:A) + (B:B) + (C:C)$. It thus appears that there are an infinity of algebras exactly analogous to quaternions, namely, those formed from the vids of the matrices whose order is prime. I shall now proceed to show that this analogy may be still farther extended, and that all the algebras analogous to quaternions, and indeed matrices of any order, admit of selective symbols like the S and V of quaternions, are resolvable into the product of a versor and a tensor, and that there are functions similar to the conjugate of a quaternion, equal in number to $\omega - 1$; if ω is the order of the algebra. I shall show this in the case of nonions, but the proof will be applicable to any case.

§21. A quaternion q may, in general, be represented by

$$q = Sq + Vq = a + bi,$$

where i is a unit vector. In matrix form this is

$$q = a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix} = \begin{pmatrix} a + b\sqrt{-1} & 0 \\ 0 & a - b\sqrt{-1} \end{pmatrix}.$$

Hence $a + b\sqrt{-1} = Sq + TVq.\sqrt{-1}$, and $a - b\sqrt{-1} = Sq - TVq.\sqrt{-1}$ are the latent roots of q.

$$Kq = Sq - Vq = \begin{pmatrix} a - b\sqrt{-1} & 0 \\ 0 & a + b\sqrt{-1} \end{pmatrix},$$

* These units are the converse of those given by the method of the preceding section. The first of the series of quadrate algebras analogous to quaternions, nonions, was discovered independently by the Peirces and Sylvester. I have in I, §10, given a short account of this discovery.

i. e. the conjugate of q is obtained from q by interchanging its latent roots but leaving its axes unchanged.

Now a nonion, or matrix of the third order, provided its axes are taken to represent the ground, may in general be put in the form

$$\begin{pmatrix} g_1 & 0 & 0 \\ 0 & g_2 & 0 \\ 0 & 0 & g_3 \end{pmatrix} = \begin{pmatrix} a+b+c & 0 & 0 \\ 0 & a+\lambda b+\lambda^2 c & 0 \\ 0 & 0 & a+\lambda^2 b+\lambda c \end{pmatrix}$$

$$= a\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + b\begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda^2 \end{pmatrix} + c\begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda^2 & 0 \\ 0 & 0 & \lambda \end{pmatrix},$$

where λ is an imaginary cube root of unity. The first of the matrices in the third member is unity, the second is a non-scalar cube root of unity, the third, being the square of the second, is also a non-scalar cube root of unity. Denoting the second matrix by i, n becomes $a+bi+ci^2$. To select the scalar and non-scalar parts of n, the selective symbols S and V may be employed; and to discriminate between the first and second parts of the non-scalar portion of n the V may be written with subscripts 1 and 2. Employing this notation, we have

$$n = Sn + V_1 n + V_2 n = a + bi + ci^2.$$

And since $V_1 n$ and $V_2 n$ are scalar multiples of non-scalar cube roots of unity, we have $V(V_1 n)^3 = V(V_2 n)^3 = 0$, just as in quaternions $V(Vq)^2 = 0$; and as in quaternions the last formula gives the important result $V(Vq.Vq' + Vq'.Vq) = 0$, so letting $V_1 n = \alpha_1$, $V_1 n' = \beta_1$, $V_1 n'' = \gamma_1$, and $V_2 n = \alpha_2$, $V_2 n' = \beta_2$, $V_2 n'' = \gamma_2$, from the two nonion formulae we may obtain in a similar way the two following results:

$$V(\alpha_1\beta_1\gamma_1 + \alpha_1\gamma_1\beta_1 + \beta_1\alpha_1\gamma_1 + \beta_1\gamma_1\alpha_1 + \gamma_1\alpha_1\beta_1 + \gamma_1\beta_1\alpha_1) = 0,$$

and a similar formula in which the α_1, β_1, γ_1 are replaced by α_2, β_2, γ_2.

As in quaternions $V^2 q = -T^2 Vq$, so in the algebra of nonions we may write

$$b^3 = V_1^3 n = T^3 V_1 n, \quad c^3 = V_2^3 n = T^3 V_2 n.$$

Since the conjugate of q is obtained by interchanging its latent roots, this suggests that a cyclic interchange of the latent roots of n, leaving its axes

unchanged, should produce a function of n similar to the conjugate of q; and the conjugate of n may be defined by the equation

$$Kn = \begin{pmatrix} g_2 & & \\ & g_3 & \\ & & g_1 \end{pmatrix} = Sn + \lambda V_1 n + \lambda^2 V_2 n.$$

The conjugate of Kn is KKn, which may be written $K^2 n$; and evidently

$$K^2 n = \begin{pmatrix} g_3 & & \\ & g_1 & \\ & & g_2 \end{pmatrix} = Sn + \lambda^2 V_1 n + \lambda V_2 n.$$

These formulae resemble that for the conjugate of q. In quaternions $K^2 = 1$, but in nonions we may write $K^2 = K^{-1}$, and $K^3 = 1$.

The tensor of a quaternion may be defined as the square root of the product of the quaternion and its conjugate. Following the analogy thus suggested, the tensor of a nonion n may be defined by the equation

$$T^3 n = n . Kn . K^2 n = a^3 + b^3 + c^3 - 3abc,$$
$$\therefore\ T^3 n = S^3 n + T^3 V_1 n + T^3 V_2 n - 3Sn . TV_1 n . TV_2 n,$$

which is the analogue of $T^2 q = S^2 q + T^2 Vq$. It is readily seen that the square of the tensor of a quaternion is equal to the product of its latent roots, and thus to its content; and similarly, that the cube of the tensor of a nonion is equal to the product of its latent roots, and hence to its content. Whence, since the determinant or content of the product of two matrices is equal to the product of their contents, the tensor of the product of two nonions is equal to the product of their tensors. When a nonion is expressed in terms of unity and eight non-scalar cube roots of unity, this proposition gives a theorem analogous to Euler's theorem when that is regarded as a theorem relating to the product of two quaternions; but the tensor of a nonion is then too complicated an expression to give the theorem any interest.

§22. If i is a nonion cube root of unity whose latent roots are $1, \lambda, \lambda^2$ (λ being an imaginary cube root of unity), and if ε denotes the base of the Naperian logarithms, by Sylvester's formula,

$$\varepsilon^{\theta i} = \tfrac{1}{3}\left[\varepsilon^{\theta}(i^2 + i + 1) + \lambda\varepsilon^{\lambda\theta}(i^2 + \lambda i + \lambda^2) + \lambda^2\varepsilon^{\lambda^2\theta}(i^2 + \lambda^2 i + \lambda)\right].$$

The coefficients of unity, i, and i^2, in this expression I shall denote by $f_0(\theta)$, $f_1(\theta)$ and $f_2(\theta)$; they are obviously analogous to $\sin\theta$ and $\cos\theta$ which appear in the corresponding expression for $\varepsilon^{\theta\alpha}$ in quaternions (where α is a unit vector). This gives

$$\begin{aligned} \varepsilon^{\theta i} &= f_0(\theta) + f_1(\theta).i + f_2(\theta).i^2, \\ \therefore\ \varepsilon^{\lambda\theta i} &= f_0(\theta) + \lambda f_1(\theta).i + \lambda^2 f_2(\theta).i^2, \\ \varepsilon^{\lambda^2\theta i} &= f_0(\theta) + \lambda^2 f_1(\theta).i + \lambda f_2(\theta).i^2. \end{aligned}$$

The second and third expressions are, severally, the first and second conjugates of $\varepsilon^{\theta i}$. Since $T^3\varepsilon^{\theta i} = \varepsilon^{\theta i}\varepsilon^{\lambda\theta i}\varepsilon^{\lambda^2\theta i} = 1$, hence

$$\overline{f_0(\theta)}^3 + \overline{f_1(\theta)}^3 + \overline{f_2(\theta)}^3 - 3f_0(\theta).f_1(\theta).f_2(\theta) = 1.$$

This suggests the corresponding formula $\cos^2\theta + \sin^2\theta = 1$. The properties of the sine and cosine, that $\cos(-\theta) = \cos\theta$, $\sin(-\theta) = -\sin\theta$, have their analogues in these functions; for from the values for $\varepsilon^{\lambda\theta i}$ and $\varepsilon^{\lambda^2\theta i}$, it follows that

$$f_0(\lambda\theta) = f_0(\theta),\quad f_1(\lambda\theta) = \lambda f_1(\theta),\quad f_2(\lambda\theta) = \lambda^2 f_2(\theta).$$

These functions also give rise to a formula in nonions analogous to De Moivre's theorem: thus, since $\varepsilon^{\alpha i}\varepsilon^{\beta i} = \varepsilon^{(\alpha+\beta)i}$, hence

$$\begin{aligned} [f_0(\alpha) + f_1(\alpha).i + f_2(\alpha).i^2][f_0(\beta) + f_1(\beta).i + f_2(\beta)i^2] \\ = f_0(\alpha+\beta) + f_1(\alpha+\beta).i + f_2(\alpha+\beta).i^2. \end{aligned}$$

Whence arise formulae for the functions of the sum of two arguments analogous to the formulae for the cosine and sine of the sum of two angles, namely,

$$\begin{aligned} f_0(\alpha+\beta) &= f_0(\alpha).f_0(\beta) + f_1(\alpha)f_2(\beta) + f_2(\alpha)f_1(\beta), \\ f_1(\alpha+\beta) &= f_0(\alpha).f_1(\beta) + f_1(\alpha).f_0(\beta) + f_2(\alpha)f_2(\beta), \\ f_2(\alpha+\beta) &= f_0(\alpha).f_2(\beta) + f_2(\alpha).f_0(\beta) + f_1(\alpha).f_1(\beta). \end{aligned}$$

It is now easy to perceive that,—just as for any quaternion q, for a proper value of θ,

$$q = \varepsilon^{\log Tq + \tan^{-1}\frac{TVq}{TSq}.\,UVq} = Tq(\cos\theta + \sin\theta.\,UVq);$$

so for any nonion n for proper values of θ and η,

$$\begin{aligned} n = \varepsilon^{\log Tn + \theta i + \eta i^2} &= Tn(f_0\theta + f_1\theta.i + f_2\theta.i^2)(f_0\eta + f_1\eta.i^2 + f_2\eta.i) \\ &= Tn\big(f_0(\theta+\eta) + f_1(\theta+\eta)i + f_2(\theta+\eta)i^2\big). \end{aligned}$$

We have therefore

$$\begin{aligned} n &= Tn.\,Un, \\ n.n' &= Tn.\,Tn'.\,Un.\,Un'. \end{aligned}$$

§23. Another analogy exists between quaternions and the quadrates of prime order, namely, that just as quaternions is linear in unity and two square roots of unity and their product, so the matrices of order ω constitute an algebra linear in unity and two ω^{th} roots of unity, their powers and products. Thus in the case of nonions, as Sylvester has shown,* if λ is an imaginary cube root of unity, and

$$i = \begin{pmatrix} 0 & 0 & 1 \\ \lambda & 0 & 0 \\ 0 & \lambda^2 & 0 \end{pmatrix}, \quad j = \begin{pmatrix} 0 & 0 & 1 \\ \lambda^2 & 0 & 0 \\ 0 & \lambda & 0 \end{pmatrix},$$

then $ij = \lambda^2 ji$ and $ji = \lambda ij$, while the products formed from the two sets $(1, i, i^2)$ and $(1, j, j^2)$ give all the nine units of the octanomial form of nonions. Moreover, I find that just as there are an infinite number of systems of three mutually normal unit vectors, so there is an infinite number of systems of eight cube roots of unity similar to the system formed from i and j and their products. The general expressions for these two new cube roots i_1 and j_1, from which the new system is to be formed, are

$$i_1 = (\varpi) \begin{pmatrix} 0 & 0 & 1 \\ \lambda & 0 & 0 \\ 0 & \lambda^2 & 0 \end{pmatrix} (\varpi^{-1}), \quad j_1 = (\varpi) \begin{pmatrix} 0 & 0 & 1 \\ \lambda^2 & 0 & 0 \\ 0 & \lambda & 0 \end{pmatrix} (\varpi^{-1}),$$

where $|\varpi| \doteq 0$. If the axes of i are (ρ_1, ρ_2, ρ_3) and the axes of j are $(\sigma_1, \sigma_2, \sigma_3)$, then the axes of i_1 and j_1 are respectively $(\varpi\rho_1, \varpi\rho_2, \varpi\rho_3)$ and $(\varpi\sigma_1, \varpi\sigma_2, \varpi\sigma_3)$. The condition that from i_1 and j_1 an octanomial system shall be formed similar to that formed from i and j, is that i_1 and j_1 shall have as latent roots the three cube roots of unity, and that the axes of i_1 and j_1 shall be related in the same way as those of i and j. If $(\alpha_1, \alpha_2, \alpha_3)$ represent the ground

$$\begin{aligned} \rho_1 &= \alpha_1 + \lambda\alpha_2 + \alpha_3, & \sigma_1 &= \alpha_1 + \lambda^2\alpha_2 + \alpha_3, \\ \rho_2 &= \alpha_1 + \alpha_2 + \lambda\alpha_3, & \sigma_2 &= \lambda^2\alpha_1 + \alpha_2 + \alpha_3, \\ \rho_3 &= \lambda\alpha_1 + \alpha_2 + \alpha_3, & \sigma_3 &= \alpha_1 + \alpha_2 + \lambda^2\alpha_3. \end{aligned}$$

With respect to the general case, let i be a primitive ω^{th} root of unity formed according to the method of §19, whose leading constituent is in the k^{th} place, and the coefficients of whose successive vids (beginning with that in the first row) are $1, \lambda, \lambda^2, \ldots \lambda^{\omega-1}$ (λ being an imaginary ω^{th} root of unity); and

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* Johns Hopkins University Circulars, No. 17 (Aug. 1882).

let j be formed from the same vids with the coefficients of i replaced respectively by their squares; then $ij = \lambda^{\omega-k+1}ji$ and $ji = \lambda^{k-1}ij$. If $k = 2$, $ij = \lambda^{\omega-1}ji$ and $ji = \lambda ij$.

Quadrate Algebras whose Order is not a Prime Number.

§24. With regard to the quadrates whose order is not a prime number, it may be shown also by a method similar to the method of §19, that they possess $\omega^2 - 1$ linearly independent non-scalar ω^{th} roots of unity, formed from their vids in the same manner in which the i, j, k of quaternions are formed from the vids of a matrix of order two. However, of these $\omega^2 - 1$ ω^{th} roots, only $(\omega + 1).\tau(\omega)$ (where $\tau(\omega)$ denotes the totient of ω) are primitive, namely, $\tau(\omega)$ ω^{th} roots formed from the vids along the diagonal, and $\omega.\tau(\omega)$ others formed from the non-symmetric vids. The latter consist of those roots in which the leading constituent is in a row whose order less one is prime to ω. The roots in which the leading constituent is in a row whose order less one has with ω the greatest common divisor ∂, other than unity, are either $(\omega : \partial)^{\text{th}}$ roots of unity, or such to a factor *près*. Of the ω^{th} roots of unity formed from the vids along the diagonal by taking as their coefficients powers of the scalar ω^{th} roots of unity, $1, \lambda, \lambda^2, \dots \lambda^{\omega-1}$, only those are primitive ω^{th} roots of which the power taken of the set of λ's is an integer prime to ω.

However, all these quadrates are analogous to quaternions in admitting of selective symbols, in having functions analogous to the conjugate in quaternions, and in that any expression in them is resolvable into the product of a tensor and a versor. This may be proved in precisely the same way in which in §21 and §22 it is shown to be true for the quadrate algebra of order three.

The quadrate algebras whose order is not a prime number are compounds of other algebras, by which I mean that they are linear in the products of the units of these algebras: if $\omega = \omega'.\omega''$, then the quadrate algebra of order ω is linear in the products of the units of the quadrate algebras of order ω' and ω'', the units of each of these systems being regarded as commutative with those of the other. Thus, the algebra formed from the vids of a matrix of order four is linear in the products of two quaternion sets, the i, j, k of the one set being commutative with the i, j, k of the other set; the algebra formed from the vids of a matrix of order six is a compound of a quaternion set and a nonion set, the units of the quaternion set being commutative with those of the nonion

set. It is obviously sufficient to prove this for any form of the several algebras, and for this purpose I choose the canonical form, and I shall show that the vids of any matrix of order ω' combined with those of any matrix of order ω'' (the vids of the first set being regarded as commutative with those of the second set), give an algebra whose multiplication table is the same as that formed from the vids of a matrix of order $\omega = \omega'.\omega''$. I shall, however, first illustrate this theorem by the case of the matrix of order four. Considering the two quadrate systems

$$\begin{array}{cccc} (A_1:A_1) & (A_1:A_2) \qquad & (B_1:B_1) & (B_1:B_2), \\ (A_2:A_1) & (A_2:A_2) \qquad & (B_2:B_1) & (B_2:B_2), \end{array}$$

which will be regarded as commutative, evidently the complete system formed by their products will consist of sixteen linearly independent units which may be arranged as follows:

$$\begin{array}{cc|cc} (A_1:A_1)(B_1:B_1) & (A_1:A_2)(B_1:B_1) & (A_1:A_1)(B_1:B_2) & (A_1:A_2)(B_1:B_2) \\ (A_2:A_1)(B_1:B_1) & (A_2:A_2)(B_1:B_1) & (A_2:A_1)(B_1:B_2) & (A_2:A_2)(B_1:B_2) \\ \hline (A_1:A_1)(B_2:B_1) & (A_1:A_2)(B_2:B_1) & (A_1:A_1)(B_2:B_2) & (A_1:A_2)(B_2:B_2) \\ (A_2:A_1)(B_2:B_1) & (A_2:A_2)(B_2:B_1) & (A_2:A_1)(B_2:B_2) & (A_2:A_2)(B_2:B_2) \end{array}$$

Those compound vids in the upper left-hand group consist of the first set, each multiplied by $(B_1:B_1)$; those in the other groups also of the first set multiplied respectively by $(B_1:B_2)$, $(B_2:B_1)$ and $(B_2:B_2)$. According to this scheme the product of the vid $(A_x:A_y)$ from the first system and the vid $(B_r:B_t)$ in the second system occupies in the resulting system a position which may be denoted by $(x + \omega'\overline{r-1},\ y + \omega'\overline{t-1})$, where, as before, the first number denotes the row and the second the column; and in this case $\omega' = \omega'' = 2$. On trial it will be found that this compound system has the same multiplication table as that of the system of vids of a matrix of order four, the vids in the compound system corresponding to those of the quadrate system of order four which occupy the same place. It should be observed that if I and I' are vids of the first quadrate system and J and J' vids of the second, that the product of the compound vids $I.J$ and $I'.J'$ is zero, unless $I.I' \neq 0$ and $J.J' \neq 0$; since the I's and J's are commutative. The proof in the general case may be accomplished as follows: Following the same scheme of arrangement of the compound vids formed from

the product of the vids of two quadrate systems of order ω' and ω'' respectively, we have

$$IJ \equiv (A_x : A_y)(B_r : B_s) = (x + \omega'\overline{r-1},\ y + \omega'\overline{s-1}) \equiv K,$$
$$I'J' \equiv (A_{y'} : A_z)(B_{s'} : B_t) = (y' + \omega'\overline{s'-1},\ z + \omega'\overline{t-1}) \equiv K'.$$

The product of IJ and $I'J'$ is zero, unless $y' = y$ and $s' = s$, when we have

$$IJ . I'J' = (A_x : A_z)(B_r : B_t),$$

which in the compound system will occupy the $(x + \omega'\overline{r-1})^{\text{th}}$ row and $(z + \omega'\overline{t-1})^{\text{th}}$ column, and thus is represented by $(x + \omega'\overline{r-1},\ z + \omega'\overline{t-1})$. The necessary and sufficient condition, however, that the compound system shall have the same law of multiplication as that of the algebra formed from the vids of a matrix of order $\omega = \omega' . \omega''$ is that KK' shall be zero unless

$$y' + \omega'\overline{s'-1} = y + \omega'\overline{s-1},$$

in which case we must have

$$KK' = (x + \omega'\overline{r-1},\ z + \omega'\overline{t-1}).$$

But since $y' \leqq \omega'$, $y \leqq \omega'$, if $y' + \omega'\overline{s'-1} = y + \omega'\overline{s-1}$, then $y' = y$ and $s' = s$. Hence the condition is fulfilled.

§25. From the last section it follows that the matrix of order $\omega = 2^m$ is a compound of m quaternion algebras which do not interfere, i. e. the units of each set are commutative with those of the other sets. I shall term such an algebra the *m-way quaternion algebra.* The system of m-way quaternion algebras has already been considered by Clifford (this Journal, Vol. I). Clifford, however, approached the subject from an entirely different point of view. He starts with n "elementary units" $i_1 i_2 \ldots i_n$ whose multiplication with each other is polar, and which are such that the square of each is -1. The 2^n products, of order 0 to n, of these elementary units, are linearly independent; and Clifford shows that the products of even order, 2^{n-1} in number (which may be obtained from all combinations of the binary products $i_1 i_2$, $i_1 i_3$, etc.), form an algebra or system by themselves, which he terms the *n-way algebra.* If $n = 2m + 1$, Clifford further proves that "the n-way algebra is a compound of m sets of quaternions which do

not interfere;" and if $n = 2m$, he shows that the n-way algebra is a compound of m quaternion sets which do not interfere, and the algebra $(1, \varepsilon)$, where $\varepsilon^2 = 1$ and ε is commutative with each of the quaternion sets. At present we are only concerned with the n-way algebra when n is odd; it consists, we have seen, of $2^{n-1} = 2^{2m}$ linearly independent units, and is a compound of m quaternion sets which do not interfere. Hence Clifford's $n (= 2m + 1)$-way algebra is that formed from the vids of a matrix of order 2^m, or is the m-way quaternion algebra. Commenting on the surprising fact that sets of quaternions should appear as the simplest form of an algebra which at first sight is so far from suggesting Hamilton's system, Clifford says that "thus it appears that quaternions is the last word of algebra to geometry." It is still more surprising that quaternions should be so prominent in the theory of matrices, and in a sense embrace the whole subject; for since the theory of matrices of any order is included in the theory of matrices of higher order, and since however great a number may be, a power of two may be obtained which is still greater, it follows that the theory of matrices of any order is included in the theory of the combination of a certain finite number of quaternion sets which do not interfere.

With regard to Clifford's geometrical algebras I am able to show that the entire system formed from the combinations of all orders of the n elementary units, which I term the n-fold algebra, is identical with the $(n - 1)$-way algebra; and when $n = 2m$, by an argument similar to that employed in the case of bi-quaternions by Benjamin Peirce, that the n-way algebra is a compound of two groups of m sets of quaternions which do not interfere, such that every set of either group is nil-facient and nil-faciend with any set of the second group.

It is clear that any quantity in Clifford's n-fold algebra may be expressed as a sum of products of expressions $\alpha = \Sigma a_1 i_1$, $\beta = \Sigma b_1 i_1$, etc., linear in the elementary units; the α, β, etc., are of course such that $\alpha^2 = -\Sigma a_1^2$, $\beta^2 = -\Sigma b_1^2$; whence, on account of the obvious analogy, they may be termed vectors. In virtue of the relation stated above which a matrix of any order holds to those of higher order, and the consequent inclusion of the theory of matrices of any order in that of sets of commutative quaternions, the theorem follows that the theory of matrices is the theory of expressions which are sums of products of quantities α, β, etc., whose squares are negative scalars.

The Identical Equation.

§26. The representation of any matrix as linear in unity and the $\omega - 1$ power of a non-scalar ω^{th} root of unity gives a very simple demonstration of the identical equation in the case of quaternions and of nonions. Thus,

$$(q - Sq)^2 = V^2 q = - T^2 Vq,$$
$$\therefore\ q^2 - 2Sq.q + T^2 q = 0,$$

$$\begin{aligned}(n - Sn)^3 &= (V_1 n + V_2 n)^3 = (bi + ci^2)^3\\ &= b^3 + c^3 + 3bc\,(bi + ci^2)\\ &= T^3 V_1 n + T^3 V_2 n + 3\,TV_1 n\, TV_2 n\,(n - Sn),\end{aligned}$$

$$\therefore\ n^3 - 3Sn.n^2 + 3\,(S^2 n + TV_1 n.TV_2 n.Sq)\,n - T^3 n = 0.$$

By the following method the identical equation is proved immediately for any case. It, however, depends upon the lemma that any symmetric function of ω conjugate matrices is a scalar, and is the same symmetric function of their latent roots. The lemma is very readily proved by considering the matrix in the form

$$g_1(\alpha_1 : \alpha_1) + g_2(\alpha_2 : \alpha_2) + \ldots . + g_\omega(\alpha_\omega : \alpha_\omega),$$

when the conjugates appear as

$$g_2(\alpha_1 : \alpha_1) + g_3(\alpha_3 : \alpha_3) + \ldots . + g_1(\alpha_\omega : \alpha_\omega),$$

etc. Returning to the identical equation, we have for $\omega = 3$,

$$\begin{aligned}0 &= (n - n)(n - Kn)(n - K^2 n)\\ &= n^3 - (n + Kn + K^2 n)\,n^2 + (Kn.K^2 n + K^2 n.n + n.Kn)\,n - n.Kn.K^2 n\\ &= n^3 - (g_1 + g_2 + g_3)\,n^2 + (g_2 g_3 + g_3 g_1 + g_1 g_2)\,n - g_1 g_2 g_3.\end{aligned}$$

The same proof applies in any case.

POSTSCRIPT.

When the above paper was written I did not know of Mr. Bucheim's paper on the *Theory of Matrices* in the Proc. London Math. Soc. (Vol. 16). Mr. Bucheim's proofs of Sylvester's law of nullity, of the identical equation, and (by implication) of the law of latency and what I have termed the law of congruity of the axes, are substantially the same as mine. I have further completed the investigation of the corollary of the law of nullity, have thoroughly treated the roots of unity and of zero, and have shown that there is an infinity of linear algebras, constituting a sequel to quaternions and nonions, whose laws of combination are equivalent to those of matrices of prime order. I have also shown that the laws of combination of matrices of composite order are identical with those of algebras whose units are the products of the units of linear algebras analogous to quaternions and nonions. I have treated the whole subject more in detail and more systematically than Mr. Bucheim.